GENERALIZED 1-SKELETA AND A LIFTING RESULT

CHRIS MCDANIEL

ABSTRACT. In [4] Guillemin and Zara described some beautiful constructions enabling them to use Morse theory on a certain class 1-skeleta including 1-skeleta of simple polytopes. In this paper we extend some of the notions and constructions from [4] to a larger class of 1-skeleta that includes 1-skeleta of projected simple polytopes. As an application of these ideas we prove a lifting result for 1-skeleta, which yields a characterization of 1-skeleta coming from projected simple polytopes.

1. INTRODUCTION

A 1-skeleton (in $\mathbb{R}^n$) is a triple consisting of a d-valent graph Γ, a *connection*, θ, that matches edges at adjacent vertices, and an *axial function*, α, that assigns pairwise independent vectors (in $\mathbb{R}^n$) to the edges issuing from each vertex of Γ satisfying certain coplanarity conditions determined by θ. The notion of a 1-skeleton appeared in the work of Goresky, Kottwitz, and MacPherson [2] as a combinatorial model of the 0- and 1-dimensional torus orbits on a certain type of topological space, now called a GKM space, from which its equivariant cohomology could be computed. Later, in a series of papers [3–5], Guillemin and Zara formulated the notion of an abstract 1-skeleton, and showed that many topological theorems regarding GKM manifolds could be interpreted as combinatorial theorems on 1-skeleta. An important example of a GKM manifold is a smooth projective toric variety, taken either with its full torus action or with a restricted subtorus action; here the 1-skeleton is that of the associated simple polytope or, for restricted actions, a linear projection of it. This paper is the outgrowth of an effort to understand the class of 1-skeleta coming from simple polytopes and their projections.

The d-valent d-independent 1-skeleton of a simple d-polytope can be built up inductively from 1-skeleta of smaller simple polytopes. In fact, one can use Morse theory on simple polytopes to show that any simple d-polytope is obtained from a d-simplex through a finite sequence of "flips". Given a simple d-polytope P, a *Morse function* for P is defined by any (generic) linear functional. The level set of a fixed Morse function at a regular value c is then a simple $(d-1)$-polytope called a *c-cross section*, realized as the intersection of P with an appropriate translate of the vanishing

hyperplane of the fixed linear functional. A flip then describes the change in the cross sections of a certain auxiliary simple $(d+1)$-polytope $\hat{P}$, whose initial cross section is the d-simplex and whose intermediate cross section is P. This Morse theory has been used before, in [6] and [7], to study the polytope algebra of a simple polytope.

Another class of 1-skeleta which should carry a Morse theory, at least in principle, are the 1-skeleta coming from linear projections of simple polytopes. In [4], Guillemin and Zara showed how to extend Morse theory to a class of 3-independent 1-skeleta called *noncyclic*, a class properly containing the simple polytopes. One of the key ideas from [4] is a beautiful construction called *reduction* which, on a (3-independent) noncyclic d-valent 1-skeleton, yields cross sections that are (2-independent) $(d-1)$-valent 1-skeleta. Unfortunately the 3-independence condition excludes many of the projected simple polytopes from the noncyclic class.

In Section 3 we introduce the class of *reducible* 1-skeleta, a class properly containing the projected simple polytopes as well as the noncyclic class. We introduce a *generalized 1-skeleton* by relaxing the pairwise independence condition on α, and keeping track of the *compatibility system*, λ, a system of positive constants related to the coplanarity conditions linking α and θ. The point then is that the reduction construction of Guillemin and Zara still yields meaningful cross sections on reducible 1-skeleta that turn out to be generalized 1-skeleta.

A reducible d-valent d-independent 1-skeleton has restrictions on its cross sections that are inherited through projection. For example, a cross section of a reducible 1-skeleton comes with two natural connections and compatibility systems, both of which agree if the 1-skeleton is d-independent. In fact the two connections on each cross section of a reducible 1-skeleton agree if and only if the 2-faces have *trivial normal holonomy*, and the two compatibility systems on each cross section agree if and only if the 2-faces are *level*. A d-valent 1-skeleton has a *total lift* if and only if it is the projection of a d-valent d-independent 1-skeleton. In Section 4 we prove the main result in this paper which characterizes projections of reducible d-valent d-independent 1-skeleta.

Theorem 1. *Let $(\Gamma, \alpha, \theta, \lambda)$ be a d-valent reducible 1-skeleton in $\mathbb{R}^n$. Then $(\Gamma, \alpha, \theta, \lambda)$ has a total lift if and only if*
(†)   *Every 2-face of $(\Gamma, \alpha, \theta, \lambda)$ is level and has trivial normal holonomy.*

In Section 2 we make the important observation that a reducible d-valent d-independent 1-skeleton in $\mathbb{R}^d$ corresponds to a complete simplicial fan in $\left(\mathbb{R}^d\right)^*$. Moreover, under this correspondence an *embedding* for the 1-skeleton corresponds to a strictly convex conewise linear function on the

fan. Thus we have the following corollary to Theorem 1, characterizing 1-skeleta of projected polytopes.

Corollary 1. *A d-valent 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ is the 1-skeleton of a projected simple polytope if and only if $(\Gamma, \alpha, \theta, \lambda)$ is reducible, all its 2-faces are level with trivial normal holonomy, and it admits an embedding $F \colon V_\Gamma \to \mathbb{R}^n$.*

In order to establish our results, we build on the machinery developed in [4], including a *blow-up* construction, a reduction construction, and a *cutting* construction. The proof of Theorem 1 follows from four lemmata. In Lemma 1 we strengthen a beautiful result in [4] that describes the change in cross sections in passing over a critical value as a blow-up/blow-down operation. Using the cutting technique from [4], Lemma 2 shows that any reducible d-valent 1-skeleton satisfying (†) can be realized as the cross section of a reducible $(d + 1)$-valent 1-skeleton satisfying (†). Lemma 3 shows that total liftability is preserved through the blow-up/blow-down operation. Finally, using Lemma 3, Lemma 4 shows that every cross section of a reducible 1-skeleton satisfying (†) necessarily has a total lift.

This paper is divided into five sections. In Section 2 we lay down some of the necessary foundational material for the theory of 1-skeleta, closely following Guillemin and Zara [4]. We also discuss the relationship between complete simplicial fans and 1-skeleta. In Section 3 we introduce the notion of a generalized 1-skeleton, and extend the necessary components of the machinery developed in [4]. In Section 4 we prove Theorem 1 and Corollary 1. In Section 5 we give some concluding remarks.

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2. 1-Skeleta

2.1. **Preliminaries.** Let Γ denote a connected regular graph with vertex set V_Γ and *oriented* edge set E_Γ, i.e. elements of E_Γ are *ordered* pairs of vertices. An oriented edge $e \in E_\Gamma$ has an *initial* vertex denoted $i(e)$, and a *terminal* vertex denoted $t(e)$. We write $e = [i(e)][t(e)]$ or if $i(e)$ and $t(e)$ are given explicitly as vertices p and q, resp., then $e = pq$. We write $\bar{e}$ to denote the oppositely oriented edge of e, i.e. if $e = pq$ then $\bar{e} = qp$. The oriented edges *at* vertex p are those edges with initial vertex p, denoted E^p. The *valency* of Γ is the cardinality of E^p, i.e. if $|E^p| = d$ we say Γ is d-valent. The first piece of structure we impose on Γ is a perfect matching of the sets $E^{i(e)}$ and $E^{t(e)}$ for all $e \in E_\Gamma$.

Definition 2.1. *A connection $\theta := \{\theta_e\}_{e \in E_\Gamma}$ on Γ is a collection of bijective maps $\theta_e: \ : E^{i(e)} \to E^{t(e)}$ indexed by the oriented edges such that $\theta_e(e) = \bar{e}$ and $\theta_e \circ \theta_{\bar{e}} = 1$ We call the pair (Γ, θ) a (d-valent) graph-connection pair.*

Next we assign vector-valued weights to the oriented edges of Γ in a way that coincides with a given connection θ on Γ.

Definition 2.2. *A function $\alpha: E_\Gamma \to \mathbb{R}^n$ is called an* axial function *for the graph connection pair (Γ, θ) if it satisfies the following axioms.*

A1. For every $p \in V_\Gamma$, the set $\{\alpha(e) \mid e \in E^p\}$ is pairwise linearly independent.

A2. For each $e \in E_\Gamma$, we have $\alpha(e) = -\alpha(\bar{e})$.

A3. For every $e \in E_\Gamma$ and each $e' \in E_{i(e)} \setminus \{e\}$ there exist positive constants $\lambda_e(e')$ such that

$$\alpha(e') - \lambda_e(e')\alpha(\theta_e(e')) \in \mathbb{R} \cdot \alpha(e).$$

The graph-connection-axial function triple is called a (*d*-valent) *1-skeleton* in $\mathbb{R}^n$, denoted $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$. The numbers $\lambda_e(e')$ form a *compatibility system* λ for the 1-skeleton. Note however that λ is completely determined by α and θ by the independence condition in A1; it is for this reason that λ is usually omitted from the notation. In our subsequent generalizations, when condition A1 is dropped, λ will enter into the notation since then keeping track of the compatibility system will be important.
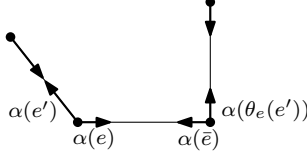

FigURE 1. An axial function on a graph

As we shall see, independence conditions on the values of an axial function at the oriented edges severely restrict the combinatorics of the 1-skeleton (Γ, α, θ).

Definition 2.3. *Given a graph Γ we say that a function $\alpha: E_\Gamma \to \mathbb{R}^n$ is*

(i) k-independent if for every vertex $p \in V_\Gamma$ and for any k-subset $e_1, \ldots, e_k$ of oriented edges at p, the set $\{\alpha(e_1), \ldots, \alpha(e_k)\}$ is linearly independent

(ii) effective if the set of vectors $\alpha(E_p) := \{\alpha(e) \mid e \in E_p\} \subset \mathbb{R}^n$ spans $\mathbb{R}^n$ for each $p \in V_\Gamma$.

We will say that the 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ is k-independent, resp. effective, if α is k-independent, resp. effective.

Note that a 1-skeleton is not an embedded graph *á priori*; the axioms do not specify position vectors for the vertices of Γ. On the other hand some embedded graphs are 1-skeleta, and conversely some 1-skeleta can be realized as embedded graphs. The following definition makes this notion precise.

Definition 2.4. *A 1 skeleton* $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ *has an* embedding *if there is a function*

$$f \colon V_\Gamma \to \mathbb{R}^n$$

with the property that for each oriented edge $pq \in E_\Gamma$ *there is a positive constant* $c_{pq} \in \mathbb{R}_+$ *such that*

$$f(q) - f(p) = c_{pq}\alpha(pq).$$

If a 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ has an embedding $f \colon V_\Gamma \to \mathbb{R}^n$, then (Γ, α, θ) can be represented as an embedded graph in the sense that V_Γ is identified with the subset $\{f(p) \mid p \in V_\Gamma\} \subset \mathbb{R}^n$ and the oriented edges $pq \in E_\Gamma$ are identified with the oriented straight line segments joining $f(p)$ to $f(q)$. In this case the axial function takes values $\alpha(pq)$ along these directed line segments $(1 - t)f(p) + tf(q)$. Although most of our examples of 1-skeleta are embedded, a general 1-skeleton need not admit an embedding at all. For example the 1-skeleton in Figure 2 is a 3-valent 3-independent 1-skeleton that does not admit an embedding. This example corresponds to a complete non-projective toric variety [1, pg. 71].
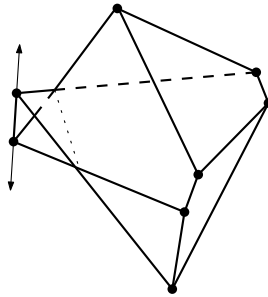

FIGURE 2. A 1-skeleton with no embedding

2.1.1. *Example: 1-skeleta from fans.* Fix a complete simplicial fan $\Sigma \subset \left(\mathbb{R}^d\right)^*$, and denote by Σ_k the set of k-cones of Σ. Let $V_\Sigma := \Sigma_d$ denote the set of d-cones, and let E_Σ denote the set of "oriented" $(d - 1)$-cones regarded as ordered intersections of d-cones, i.e. $\tau = \sigma_1 \cap \sigma_2$ and $\bar\tau = \sigma_2 \cap \sigma_1$. The pair (V_Σ, E_Σ) defines a d-valent graph Γ_Σ. For each oriented $(d - 1)$-cone $\tau = \sigma_1 \cap \sigma_2$ choose and fix a normal vector $\alpha_\tau \in \mathbb{R}^d$ pointing inside σ_1, i.e.

such that $x(\alpha_\tau) > 0$ for all $x \in \text{int}(\sigma_1)$. We can even choose our normal vectors so that $\alpha_\tau = -\alpha_{\bar{\tau}}$. The claim is that the function

$$E_\Sigma \xrightarrow{\quad \alpha \quad} \mathbb{R}^d$$

$$\tau \longmapsto \alpha_\tau$$

is a d-independent axial function on Γ_Σ. Clearly α is d-independent since the normal vectors of a d-cone are a basis for $\mathbb{R}^d$. To see that A3 holds fix an oriented $(d-1)$-cone $\tau = \sigma_1 \cap \sigma_2$ and any other "oriented edge" issuing from σ_1, say $\tau' = \sigma_1 \cap \sigma_2'$. Let ρ' and ρ'' be rays (i.e. one-dimensional cones) such that $\tau + \rho^{(i)} = \sigma_i$ for $i = 1, 2$. Then we have $\tau' = \tau \cap \tau' + \rho'$, and there is a unique $(d-1)$-cone defined by $\tau'' = \tau \cap \tau' + \rho''$. Define the connection map $\theta_\tau(\tau') := \tau''$. The claim is that there are positive constants $\lambda_\tau(\tau')$ such that

$$(2.1) \qquad\qquad \alpha_{\tau'} - \lambda_\tau(\tau')\alpha_{\tau''} = c \cdot \alpha_\tau.$$

Indeed the LHS of Eq. (2.1) necessarily vanishes on $\tau' \cap \tau'' = \tau \cap \tau'$. Let ρ be the ray determined by $\tau = \tau \cap \tau' + \rho$, and choose any nonzero covector $\eta \in \rho$. Then defining

$$(2.2) \qquad\qquad \lambda_\tau(\tau') := \frac{\langle \eta, \alpha_{\tau'} \rangle}{\langle \eta, \alpha_{\tau''} \rangle}$$

forces LHS of Eq. (2.1) to vanish on ρ as well, hence on all of τ. Thus LHS of Eq. (2.1) must be a multiple of α_τ. Thus the triple $(\Gamma_\Sigma, \alpha, \theta)$ defines a d-valent, d-independent 1-skeleton in $\mathbb{R}^d$.

A *conewise-linear function* on Σ is a continuous function $F \colon |\Sigma| \to \mathbb{R}$ whose restriction to every cone in Σ is linear. Write F_σ for the linear function that F restricts to on σ. F is called *strictly convex* if for any two distinct cones $\sigma, \sigma' \in \Sigma$ and any $x \in \sigma$ we have $F_{\sigma'}(x) > F_\sigma(x)$. A standard result states that a complete (simplicial) fan Σ admitting a strictly convex conewise-linear function $F \colon |\Sigma| \to \mathbb{R}$ is the normal fan of a convex (simple) polytope P given as the convex hull of the points F_σ. For this reason, a complete fan that admits a strictly convex conewise-linear function is called *polytopal*. Note that for any oriented edge $\tau = \sigma_1 \cap \sigma_2$ we have

$$(2.3) \qquad\qquad F_{\sigma_2} - F_{\sigma_1} = c_\tau \alpha_\tau$$

for some *positive* scalar c_τ. Indeed since F is continous, F_{σ_2} and F_{σ_1} must agree on τ, hence the LHS of Eq. (2.3) is some multiple of α_τ. By the strict convexity condition, for $x \in \sigma_1 \setminus \tau$, we have $F_{\sigma_2}(x) - F_{\sigma_1}(x) > 0$ hence the LHS is necessarily a positive multiple of α_τ. Thus a strictly convex conewise-linear function on Σ defines an embedding for $(\Gamma_\Sigma, \alpha, \theta)$.

2.2. **Subskeleta and Holonomy.** Fix a d-valent 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$.

Let p and q be vertices of Γ. A *path* from p to q is a sequence of vertices beginning with p and ending with q such that any two consecutive vertices in the path form an edge of Γ; a path from p to q is denoted by

$$\gamma \colon p \to \cdots \to q.$$

A *subgraph* of Γ is a regular graph $\Gamma_0 = (V_0, E_0)$ where $V_0 \subset V_\Gamma$ and $E_0 \subset E_\Gamma$. If the connection on Γ restricts to Γ_0 in the sense that $\theta_e \left(E_0^{i(e)} \right) \subseteq \left(E_0^{t(e)} \right)$ for every edge $e \in E_0$ we say the subgraph is *totally geodesic*. The restriction of the connection θ and axial function α to the totally geodesic subgraph Γ_0 defines a 1-skeleton triple $(\Gamma_0, \alpha_0, \theta_0)$ called a *subskeleton*.

The 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ always admits a certain class of subskeleta called *k-slices*. For each k-dimensional subspace $H \subset \mathbb{R}^n$ define Γ_H to be any connected component of the graph $G = (V_G, E_G)$ determined by the condition that the edges' directions lie in H, i.e. $E_G := \{e \in E_\Gamma \mid \alpha(e) \in H\}$ and $V_G := \{p \in V_\Gamma \mid p = i(e), \ e \in E_G\}$. The claim is that the subgraph Γ_H has constant valency and is totally geodesic. Indeed fix $e := pq$ an oriented edge of Γ_H, and let E_H^p denote the oriented edges at p lying in Γ_H. Let $e' \in E_H^p$ be any oriented edge at p different from e. Then $\alpha(\theta_e(e'))$ must lie in the 2-plane generated by $\alpha(e)$ and $\alpha(e')$ by condition A3 in Definition 2.2. Since $\alpha(e)$ and $\alpha(e')$ both lie in the subspace H, $\alpha(\theta_e(e'))$ must also lie in H, hence $\theta_e(e') \in E_H^q$. This shows that $\theta_e \left(E_H^p \right) \subseteq E_H^q$, and a similar argument shows that $\theta_{\bar{e}} \left(E_H^q \right) \subseteq E_H^p$. Since these maps are inverses of one another we see that $\left| E_H^p \right| = \left| E_H^q \right|$, and, since Γ_H is connected, it must therefore have constant valency. The k-faces of a simple polytope are k-valent k-slices of its corresponding 1-skeleton. Note that a k-slice of a 1-skeleton need not be k-valent in general.

Fix a subskeleton $(\Gamma_0, \alpha_0, \theta_0) \subseteq (\Gamma, \alpha, \theta)$. An oriented edge not in E_0 but whose initial vertex lies in V_0 is *normal* to Γ_0. The set of oriented normal edges to Γ_0 will be denoted by N_0. For each vertex $p \in V_0$ the oriented edge set E^p is partitioned into two pieces: the oriented edges at p in Γ_0, denoted E_0^p, and those at p normal to Γ_0, denoted N_0^p. Moreover, since Γ_0 is totally geodesic the connection also splits: $\theta = \theta_0 \sqcup \theta^\perp$. The collection of maps $\theta^\perp := \left\{ \theta_{pq}^\perp \colon N_0^p \to N_0^q \mid pq \in E_0 \right\}$ define the *normal connection* for the subskeleton.

Given any vertices $p, q \in V_0$ and a path joining them in Γ_0 $\gamma \colon p = p_0 \to \cdots \to p_j = q$ define the *path-connection map* for γ to be the composition

$$K_\gamma := \theta_{p_{j-1} p_j} \circ \ldots \circ \theta_{p_0 p_1} \colon E_0^{p_0} \to E_0^{p_j}.$$

Define the *normal path-connection map* for γ to be the corresponding compositon of normal connection maps:

$$K_\gamma^\perp := \theta_{p_{j-1}p_j}^\perp \circ \ldots \circ \theta_{p_0 p_1}^\perp : N_0^{p_0} \to N_0^{p_j}.$$

For γ a loop, the (resp. normal) path-connection map K_γ gives a permutation of the set (resp. N_0^p) E_0^p, called a (resp. *normal*) *holonomy map*.

Definition 2.5. *A subskeleton* $(\Gamma_0, \alpha_0, \theta_0) \subseteq (\Gamma, \alpha, \theta)$ *has* trivial normal holonomy *if the holonomy map $K_\gamma^\perp$ is trivial for all loops $\gamma \subset \Gamma_0$.*

Remark. *The k-faces of a simple polytope have trivial normal holonomy. Indeed a k-face Γ_0 containing vertex p is a k-slice for the k-dimensional subspace spanned by the set $\alpha(E_0^p)$. Then for any loop $\gamma\colon p \to p_1 \to \cdots \to p_N \to p$ in Γ_0, and for any edge $e \in N_0^p$, condition A3 implies that $\alpha(e)$ and $\alpha(K_\gamma^\perp(e))$ together with the spanning set $\alpha(E_0^p)$ span a $(k+1)$-dimensional subspace. Thus by d-independence of the set $\alpha(E^p)$ we must have $K_\gamma^\perp(e) = e$.*

2.3. **Polarizations and Combinatorial Betti Numbers.** An *orientation* of Γ is a choice of one orientation for each edge in Γ; the corresponding oriented edge is called a *directed edge* for the orientation. A path

$$\gamma\colon p \to \cdots \to q$$

is said to be *oriented* (with respect to a given orientation on Γ) if $p_i p_{i+1}$ is a directed edge for all i. The orientation is called *acyclic* if there are no oriented loops.

A covector $\xi \in (\mathbb{R}^n)^*$ is *generic* for the 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ if it pairs with each edge direction nonzero, i.e. $\langle \xi, \alpha(e) \rangle \neq 0$ for each $e \in E_\Gamma$. A generic covector ξ for (Γ, α, θ) induces an orientation on Γ by declaring the directed edges to be those oriented edges that pair positively with ξ.

Definition 2.6. *The generic covector ξ is called* polarizing *if it induces an acyclic orientation on Γ. The 1-skeleton (Γ, α, θ) admits a polarization if it admits a generic polarizing covector ξ.*

Note that a generic polarizing covector for (Γ, α, θ) is also generic and polarizing for any subskeleton $(\Gamma_0, \alpha_0, \theta_0)$.

Remark. *In [4], Guillemin and Zara use the term "polarizing" to describe what we call "generic" and what we call a "polarizing covector" they call a "polarizing covector satisfying the 'no-cycle condition'".*

A 1-skeleton need not admit any polarization at all. See Figure 3.

On the other hand if a 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ admits an embedding, then every generic covector $\xi \in (\mathbb{R}^n)^*$ is polarizing. Indeed an oriented
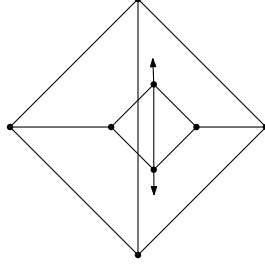
FIGURE 3. No Polarization

"embedded loop" $\gamma\colon f(p_0) \to f(p_1) \to \cdots \to f(p_N) \to f(p_0)$ in (Γ, α, θ) would give a system of inconsistent inequalities:

$$\langle \xi, f(p_0) \rangle > \langle \xi, f(p_1) \rangle > \cdots > \langle \xi, f(p_N) \rangle > \langle \xi, f(p_0) \rangle.$$

Guillemin and Zara also make this observation in [3].

Definition 2.7. *Given a polarizing covector $\xi \in (\mathbb{R}^n)^*$ for (Γ, α, θ) we say an injective function $\phi\colon V_\Gamma \to \mathbb{R}$ is a* Morse function *on (Γ, α, θ) compatible with ξ if for each edge $pq \in E_\Gamma$ satisfying $\langle \xi, \alpha(pq) \rangle > 0$ we have $\phi(p) < \phi(q)$.*

As pointed out in [4], the existence of a polarizing covector guarantees the existence of a compatible Morse function. Indeed just define $\phi(p)$ to be the length of the longest oriented path in Γ that starts at p. This is well defined since there are no oriented loops. We can then perturb ϕ a little to make it injective.

Definition 2.8. *For $p \in V_\Gamma$ define the* index *of p (with respect to a generic covector ξ) to be the number of oriented edges "flowing into" p; in other words*

$$\mathrm{ind}_\xi(p) := \#\{e \in E_p \mid \langle \xi, \alpha(e) \rangle < 0\}.$$

Define the ith combinatorial Betti number of Γ by

$$b_i(\Gamma, \alpha) := \#\{p \in V_\Gamma \mid \mathrm{ind}_\xi(p) = i\}.$$

A vertex $p_0 \in V_\Gamma$ with $\mathrm{ind}_\xi(p_0) = 0$ is called a (ξ-) source of Γ.

While the index of a vertex of Γ clearly depends on the generic covector, ξ, an elegant argument due to Bolker [4, Theorem 1.3.1] shows that the combinatorial Betti numbers are actually independent of ξ.

Guillemin and Zara [4] studied a class of 3-independent 1-skeleta they call *noncyclic*. Here is their definition:

Definition 2.9. *A 3-independent 1-skeleton $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ is called* non-cyclic *if the following conditions hold:*

NC1. $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ *admits a polarization*
NC2. $b_0(\Gamma_H, \alpha_H) = 1$ *for every* 2-slice $(\Gamma_H, \alpha_H, \theta_H)$.

The class of 1-skeleta coming from simple polytopes (or complete simplicial polytopal fans) is certainly noncyclic; indeed any generic covector is polarizing since the 1-skeleton has an embedding. It is unclear whether or not every 1-skeleton coming from a complete simplicial fan should admit a polarization. On the other hand it is clear that any generic covector for the 1-skeleton of a fan should always yield a unique source corresponding to the unique cone containing the covector.

2.3.1. *Example: Fans from 1-skeleta.* Let $\Sigma \subset \left(\mathbb{R}^d\right)^*$ be a complete simplicial d-dimensional fan with associated d-valent d-independent 1-skeleton $(\Gamma_\Sigma, \alpha, \theta) \subset \mathbb{R}^d$. Then for any genric covector $\xi \in \left(\mathbb{R}^d\right)^*$ there is a unique vertex $p_0 \in V_\Gamma$ such that $\langle \xi, \alpha(e) \rangle > 0$ for all $e \in E^{p_0}$, i.e. $b_0(\Gamma_\Sigma, \alpha) = 1$. A 1-skeleton with zeroth combinatorial Betti number equal to one will be called *pointed*. See Figure 4. In fact a stronger property holds for $(\Gamma_\Sigma, \alpha, \theta)$: any k-slice is pointed, corresponding to the fact that the link of any face of a complete simplicial fan is identified with a complete simplicial fan via orthogonal projection. A d-valent d-independent 1-skeleton will be called *toral* if every k-slice for $1 \leq k \leq d$ is pointed. Hence there is a correspondence between complete simplicial fans and toral 1-skeleta
(2.4)

$$\left\{\text{Complete simplicial fans in } \left(\mathbb{R}^d\right)^*\right\} \longrightarrow \left\{\text{Toral 1-skeleta in } \mathbb{R}^d\right\}$$

$$\Sigma \longmapsto (\Gamma_\Sigma, \alpha, \theta).$$

The claim is that (2.4) is one-to-one.

Let (Γ, α, θ) denote a toral 1-skeleton in $\mathbb{R}^d$. For each vertex p, define the simplicial polyhedral cone

$$X_p := \left\{u \in \left(\mathbb{R}^d\right)^* \mid \langle u, \alpha(e) \rangle \geq 0, \ e \in E^p\right\}.$$

Let Σ denote the set of cones X_p and all their faces. In order to establish that Σ is a complete simplicial fan in $\left(\mathbb{R}^d\right)^*$ we need to show that

   (i) $\displaystyle\bigcup_{p \in V_\Gamma} X_p = \left(\mathbb{R}^d\right)^*$
  (ii) $X_p \cap X_q$ is a face of both X_p and X_q.

Condition (i) holds since (Γ, α, θ) is a pointed d-slice by assumption. To verify (ii), fix vertices p and q, and let τ be the smallest face of X_p containing $X_p \cap X_q$. Define the k-dimensional subspace $H = \tau^\perp := \left\{v \in \mathbb{R}^d \mid \langle u, v \rangle = 0 \ \forall \ u \in \tau\right\}$. Let Γ_H^p denote the graph of the k-slice containing p and let Γ_H^q denote the

graph of the k-slice containing q. The claim is that $\Gamma_H^p = \Gamma_H^q$. Otherwise, each graph is a k-slice, and as such is pointed by a common generic covector $\xi \in H^*$. Let $p_0 \in V_H$ and $q_0 \in V_H'$ be the respective ξ-sources. Note that there exists a covector $\eta \in \tau$ such that $\langle \eta, \alpha(e) \rangle > 0$ for all $e \in N_H^{p_0} \cup N_H^{q_0}$ (otherwise there would be at least one $e \in N_H^{p_0} \cup N_H^{q_0}$ such that $\langle \eta, \alpha(e) \rangle = 0$ for all $\eta \in \tau$). Then extending ξ by zero to a covector in $\left(\mathbb{R}^d \right)^*$, and adding to it the covector η, we get a generic covector $\omega = \xi + \eta$ such that $\langle \omega, \alpha(e) \rangle > 0$ for all $e \in E^{p_0} \cup E^{q_0}$, which is impossible since $(\Gamma, \alpha, \theta) \subset \mathbb{R}^d$ is pointed. Therefore $\Gamma_H^p = \Gamma_H^q = \Gamma_H$ is a single k-slice containing both p and q.

Now we want to show that $\tau \subseteq X_p \cap X_q$. Fix $u \in \tau$. Then $\langle u, \alpha(e) \rangle = 0$ for all $e \in E_H^p$, and $\langle u, \alpha(e') \rangle > 0$ for all $e \in N_H^p$. For $pp_1 \in E_H$ note that A3 yields $\langle u, \alpha(e) \rangle = \lambda_{pp_1}(e)\langle u, \alpha(\theta_{pp_1}(e)) \rangle$. Therefore, since Γ_H is connected, we conclude that $\langle u, \alpha(e'') \rangle > 0$ for all $e'' \in N_H^q$. Thus $u \in X_p \cap X_q$, hence $X_p \cap X_q$ is a face of X_p. A similar argument shows that $X_p \cap X_q$ is also a face of X_q, hence Σ is a complete simplicial d-dimensional fan.

Note that noncyclic d-valent, d-independent 1-skeleta are necessarily toral. In particular, the one-to-one correspondence in (2.4) restricts to a one-to-one correspondence between complete simplicial polytopal fans in $(\mathbb{R}^d)^*$ and d-valent d-independent noncyclic 1-skeleta admitting embeddings.
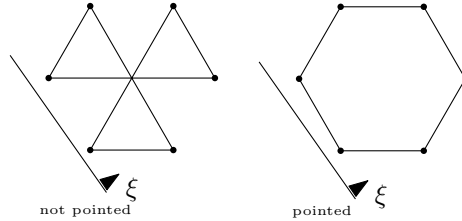

FIGURE 4. 2-valent 2-independent 1-skeleta

3. GENERALIZATIONS

3.1. Generalized 1-Skeleta. Fix a d-valent graph connection pair (Γ, θ).

Definition 3.1. *A* compatibility system *for (Γ, θ) $\lambda := \{\lambda_e\}_{e \in E_\Gamma}$ is a collection of maps $\lambda_e \colon E_{i(e)} \to \mathbb{R}_+$ indexed by the oriented edges of Γ that satisfy the rule*

$$\lambda_{\bar{e}} \circ \theta_e = \frac{1}{\lambda_e}$$

for every $e \in E_\Gamma$.

The triple $(\Gamma, \theta, \lambda)$ defines a *pre 1-skeleton*.

Definition 3.2. *A* generalized axial function α *for the pre-1-skeleton* $(\Gamma, \theta, \lambda)$ *is a map* $\alpha\colon E_\Gamma \to \mathbb{R}^n$ *that satisfies the following axioms:*

gA1. For each $e \in E_\Gamma$ *there is some* $m_e > 0$ *such that* $\alpha(e) = -m_e \alpha(\bar{e})$

gA2. For every $e \in E_\Gamma$ *and each* $e' \in E^{i(e)} \setminus \{e\}$ *we have*

$$\alpha(e') - \lambda_e(e')\alpha(\theta_e(e')) = \mathbb{R} \cdot \alpha(e).$$

The quadruple $(\Gamma, \alpha, \theta, \lambda)$ is a *generalized 1-skeleton*. Note that if the generalized axial function α is 2-independent the compatibility system λ is uniquely determined. If, in addition, the constants m_e are all equal to 1 in condition gA1 then α is actually an axial function and the quadruple $(\Gamma, \alpha, \theta, \lambda)$ defines a 1-skeleton.

It will be useful to have a notion of equivalence of generalized 1 skeleta.

Definition 3.3. *Two generalized 1 skeleta* $(\Gamma, \alpha, \theta, \lambda) \subset \mathbb{R}^n$ *and* $(\tilde{\Gamma}, \tilde{\alpha}, \tilde{\theta}, \tilde{\lambda}) \subset \mathbb{R}^n$ *are equivalent if the graph connection pairs* (Γ, θ) *and* $\Gamma', \theta')$ *are equal and there exists a function* $\kappa\colon E_\Gamma \to \mathbb{R}_+$ *such that for every* $e \in E_\Gamma$ *and* $e' \in E^{i(e)}$ *we have*

(i) $\lambda_e(e') = \dfrac{\kappa(e')}{\kappa(\theta_e(e'))}\tilde{\lambda}_e(e'),$

(ii) $\alpha(e) = \kappa(e) \cdot \alpha'(e).$

We will denote equivalence of generalized 1-skeleta by

$$(\Gamma, \alpha, \theta, \lambda) \equiv (\tilde{\Gamma}, \tilde{\alpha}, \tilde{\theta}, \tilde{\lambda})$$

Note that a 2-independent generalized 1-skeleton is equivalent to a 1-skeleton. Indeed fix any orientation of Γ, and let E_Γ^+ denote the oriented edges oriented positively. Then define $\kappa(e) = \begin{cases} 1 & \text{if } e \in E_\Gamma^+ \\ \frac{1}{m_e} & \text{if } e \notin E_\Gamma^+ \end{cases}$. We will abuse the notation slightly and call a generalized 1-skeleton with a 2-independent axial function a 1-skeleton.

The notions of subskeleton, (resp. normal) path-connection map, and (resp. normal) holonomy have obvious generalized analogues. Additionally, the compatibility system of a generalized 1-skeleton allows one to assign positive scalars to (resp. normal) path connection maps. Fix a d-valent generalized 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ and a k-valent subskeleton $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$.

Definition 3.4. *Let* $\gamma\colon p = p_0 \to p_1 \to \cdots p_{j-1} \to p_j = q$ *be a path in* Γ_0 *joining vertices* p *and* q *in* V_0. *The* path-connection number *for* γ *is the product*

$$|K_\gamma| := \left(\prod_{e \in E_0^{p_0}} \lambda_{p_0 p_1}(e)\right) \cdots \left(\prod_{e \in E_0^{p_{j-1}}} \lambda_{p_{j-1}p_j}(e)\right).$$

The normal path-connection number *for γ is the product*

$$\left|K_\gamma^\perp\right| := \left(\prod_{e \in N_0^{p_0}} \lambda_{p_0 p_1}(e)\right) \cdots \left(\prod_{e \in N_0^{p_{j-1}}} \lambda_{p_{j-1} p_j}(e)\right).$$

For each $e \in E^p$ the local path-connection number *for γ at e is the product*

$$\left|K_\gamma(e)\right| := \prod_{i=1}^{j} \lambda_{p_{i-1} p_i}(\theta_{p_{i-2} p_{i-1}} \circ \cdots \circ \theta_{p_0 p_1}(e)).$$

If γ is a loop, replace the term "path-connection" with the term "holonomy".

Of particular interest will be those subskeleta whose local normal holonomy numbers are trivial. More precisely,

Definition 3.5. *A subskeleton $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$ of a generalized 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ is* level *if for each $e \in N_0^p$ and every loop γ in Γ_0 such that $K_\gamma(e) = e$, we have $\left|K_\gamma(e)\right| = 1$.*

Remark. *Note that the k-faces of a simple polytope are level. Indeed for face F and a fixed basepoint $p_0 \in F$, let e be any edge normal to F at p_0. Then there is a unique facet H containing F for which e is normal, and H corresponds to a $(d-1)$-slice $(\Gamma_H, \alpha_H, \theta_H)$. Then for any choice of nonzero covector $\eta \in \left(\mathbb{R}^d\right)^*$ normal to the hyperplane $\mathrm{span}(H) \subset \mathbb{R}^d$, the compatibility constants around the edges of Γ_H are given by the formula $\lambda_{pq}(e) = \dfrac{\langle \eta, \alpha(e) \rangle}{\langle \eta, \alpha(\theta_{pq}(e)) \rangle}$ as in Eq. (2.2). Thus the local normal holonomy numbers associated to a loop $\gamma\colon p_0 \to p_1 \to \cdots \to p_N \to p_0$ are given by*

$$\left|K_\gamma^\perp(e)\right| = \frac{\langle \eta, \alpha(e) \rangle}{\langle \eta, \alpha(\theta_{p_0 p_1}(e)) \rangle} \cdots \frac{\langle \eta, \alpha(\theta_{p_{N-1} p_N} \circ \cdots \circ \theta_{p_0 p_1}(e)) \rangle}{\langle \eta, \alpha(K_\gamma^\perp(e)) \rangle} = 1.$$

The notions of polarizations, compatible Morse functions, and combinatorial Betti numbers also have obvious generalized analogues.

3.2. **Blow-Up.** Fix a d-valent generalized 1-skeleton with connection $(\Gamma, \alpha, \theta, \lambda) \subset \mathbb{R}^n$ and let $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$ be a k-valent subskeleton. We set up the following notation as a convention to be used throughout this section. Let $p, q, r \in V_0$ denote arbitrary vertices with $pq, pr \in E_0$. Let $e, e', e'' \in N_0^p$ denote arbitrary edges at p normal to Γ_0 and set $f, f', f'' \in N_0^q$, and $g, g', g'' \in N_0^r$ such that $\theta_{pq}(e^{(i)}) = f^{(i)}$ and $\theta_{pr}(e^{(i)}) = g^{(i)}$ for $0 \le i \le 2$. For a subset of oriented edges $E \subset E_\Gamma$ set $\overline{E} := \left\{\overline{e} \mid e \in E\right\}$.

We will define a new graph $\Gamma^\sharp = (V^\sharp, E^\sharp)$ by cutting off Γ_0 in Γ and replacing it with a new $(d-1)$-valent sub-graph. Define the "new vertex set" $V_0^\sharp := \left\{ z_e^p \mid p \in V_0, \ e \in N_0^p \right\}$. Define the vertex set of $\Gamma^\sharp$ to be

$$V^\sharp := V_\Gamma \setminus V_0 \sqcup V_0^\sharp.$$

Define the "new edge sets" $N_0^\sharp := \left\{ z_e^p[t(e)] \middle| p \in V_0, \ e \in N_0^p \right\}$ and $E_0^\sharp := \left\{ z_e^p z_{e'}^p \right\} \cup \left\{ z_e^p z_f^q \right\}$. Define the edge set of $\Gamma^\sharp$ by

$$E^\sharp = E_\Gamma \setminus \left(E_0 \cup N_0 \cup \overline{N}_0 \right) \cup N_0^\sharp \cup \overline{N}_0^\sharp \cup E_0^\sharp.$$

Note that oriented edge sets $N_0^\sharp$ and $\overline{N}_0^\sharp$ are in one-to-one correspondence with N_0 and $\overline{N}_0$. Thus it is clear that the vertices $x \in V_\Gamma \setminus V_0 \subseteq V^\sharp$ (including vertices $x = [t(e)]$, $e \in N^0$) are incident to exactly d oriented edges. Also the vertices $z_e^p \in V_0^\sharp \subseteq V^\sharp$ are incident to the d edges $\left\{ z_e^p z_{e'}^p \mid e' \in N_0^p \setminus \{e\} \right\} \sqcup \left\{ z_e^p z_f^q \mid pq \in E_0 \right\} \sqcup \left\{ z_e^p[t(e)] \right\}$. Hence $\Gamma^\sharp$ is a d-valent graph called the *blow-up graph* of Γ along Γ_0.

The natural map of sets $\beta \colon V^\sharp \to V_\Gamma$

$$\beta(x) = \begin{cases} q & \text{if } x = q \in V_\Gamma \setminus V_0 \\ p & \text{if } x = z_e^p \text{ for some } e \in N_p^0 \end{cases}$$

extends to a map of graphs $\beta \colon \Gamma^\sharp \to \Gamma$ called the *blow down map*.

The induced subgraph of $\Gamma^\sharp$ on the vertex set $\beta^{-1}(V_0) = V_0^\sharp$, denoted by $\Gamma_0^\sharp$, is a $(d-1)$-valent connected subgraph called the *singular locus* of the blow-up.
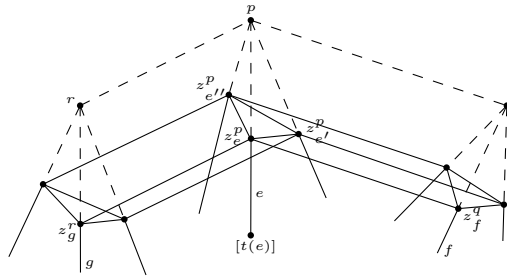

FIGURE 5. blow-up along a sub-skeleton

For each vertex x in the singular locus $V_0^\sharp$ define the *horizontal edges at x* to be the edges in the singular locus at x preserved by β, i.e. $(E_0^\sharp)_x^h := \beta^{-1}(E_{\beta(x)}^0)$. Define the *vertical edges at x* to be those edges destroyed by β, i.e. $(E^\sharp)_x^v := \left(E_0^\sharp \right)_x \setminus (E_0^\sharp)_x^h$.

We want to define a connection and compatibility system on $\Gamma^\sharp$. To avoid confusion, we will use the letter ϵ to denote an edge in $\Gamma^\sharp$ and reserve the letter e for an edge in Γ. For oriented edges ϵ not issuing from the singular locus $\Gamma_0^\sharp$, define

$$(3.1) \qquad \theta_\epsilon^\sharp := \beta^{-1} \circ \theta_{\beta(\epsilon)} \circ \beta$$

$$(3.2) \qquad \lambda_\epsilon^\sharp := \lambda_{\beta(\epsilon)} \circ \beta.$$

The values of $\theta^\sharp$ and $\lambda^\sharp$ on oriented edges in $N_0^\sharp$ and $E_0^\sharp$ are listed in Tables 1 and 2. From Eqs. (3.1) and (3.2) as well as Tables 1 and 2, it is straight forward to check that the triple $(\Gamma^\sharp, \theta^\sharp, \lambda^\sharp)$ defines a pre 1-skeleton.

ϵ \\ ϵ'	$z_e^p[t(e)]$	$z_e^p z_f^q$	$z_e^p z_{e'}^p$
$z_e^p[t(e)]$	-	$z_f^q[t(f)]$	$z_{e'}^p[t(e')]$
$z_e^p z_g^r$	$\beta^{-1} \circ \theta_e(pr)$	$z_f^q z_h^s$	$z_{e'}^p z_{g'}^r$
$z_e^p z_{e''}^{\bar{p}}$	$\beta^{-1} \circ \theta_e(e'')$	$z_f^q z_{f''}^{\bar{q}}$	$z_{e'}^p z_{e''}^{\bar{p}}$

TABLE 1. Values of $\theta_\epsilon(\epsilon')$

ϵ \\ ϵ'	$z_e^p[t(e)]$	$z_e^p z_f^q$	$z_e^p z_{e'}^p$
$z_e^p[t(e)]$	-	1	1
$z_e^p z_g^r$	$\lambda_e(pr)$	$\lambda_{pq}(pr)$	1
$z_e^p z_{e''}^{\bar{p}}$	$\lambda_e(e'')$	$\lambda_{pq}(e'')$	1

TABLE 2. Values of $\lambda_\epsilon(\epsilon')$

On oriented edges not in the singular locus of $\Gamma^\sharp$, define

$$(3.3) \qquad \alpha^\sharp := \alpha \circ \beta$$

In order to extend the function $\alpha^\sharp$ to the remaining oriented edges issuing from the singular locus, we must assume the existence of a system of positive scalar assignments to the normal edges of Γ_0, $n \colon N^0 \to \mathbb{R}_+$, satisfying the following compatibility condition:

$$(3.4) \qquad \frac{n(e')}{n(\theta_e(e'))} = \lambda_e(e') \ \text{ for all } e \in \Gamma_0 \text{ and all } e' \in N_0^{i(e)};$$

such a system is called a *blow-up system* for the subskeleton $(\Gamma_0, \alpha_0, \theta_0, \lambda_0) \subseteq (\Gamma, \alpha, \theta, \lambda)$. It is worth remarking that the levelness of $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$ guarantees the existence of a blow-up system. Indeed fixing a basepoint $p_0 \in V_0$ and setting $n(e) \equiv 1$ for all $e \in N_0^{p_0}$, one can use the normal path connection and the normal path connection numbers to extend n to N_0^x for any other

$x \in V_0$: Simple take any path $\gamma \colon p_0 \to \cdots \to x$ joining p_0 to x in Γ_0. Then for any $e \in N_0^x$ there is $\tilde{e} \in N_0^{p_0}$ such that $e = K_\gamma^\perp(\tilde{e})$. Thus the function

$$n(e) := \frac{1}{\left|K_\gamma^\perp(\tilde{e})\right|}$$

is independent of the path γ, and hence defines the required blow-up system.

We use a fixed blow-up system, $n \colon N^0 \to \mathbb{R}_+$, for $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$ to define $\alpha^\sharp$ on $E_0^\sharp \sqcup N_0^\sharp$ as shown in Table 3.

ϵ	$\alpha^\sharp(\epsilon)$
$z_e^p[t(e)]$	$\frac{1}{n(e)}\alpha(e)$
$z_e^p z_f^q$	$\alpha(pq)$
$z_e^p z_{e'}^p$	$\alpha(e') - \frac{n(e')}{n(e)}\alpha(e)$

TABLE 3. Values of $\alpha^\sharp(\epsilon)$

Clearly $\alpha^\sharp$ satisfies gA1 in Definition 3.2. It remains to show that $\alpha^\sharp$ satisfies gA2:

$$(3.5) \qquad \alpha^\sharp(\epsilon') - \lambda_\epsilon^\sharp(\epsilon')\alpha^\sharp(\theta_\epsilon^\sharp(\epsilon')) \equiv 0 \mod \alpha^\sharp(\epsilon).$$

It is straight forward to verify gA2 for oriented edges not issuing from the singular locus. Indeed in this case Eq. (3.5) becomes

$$\alpha(\beta(\epsilon') - \lambda_{\beta(\epsilon)}(\beta(\epsilon')))\alpha(\theta_{\beta(\epsilon)}(\beta(\epsilon'))) = c\alpha(\beta(\epsilon)),$$

which holds since α is a generalized axial function for the pair (Γ, θ). Thus it suffices to verify Eq. (3.5) for oriented edges issuing from the singular locus.

For $\epsilon = z_e^p z_f^q$ and $\epsilon' = z_e^p z_{e'}^p$ the LHS of Eq. (3.5) becomes

$$(3.6) \qquad \left(\alpha(e') - \frac{n(e')}{n(e)}\alpha(e)\right) - \lambda_{pq}(e') \cdot \left(\alpha(f') - \frac{n(f')}{n(f)}\alpha(f)\right).$$

Regrouping terms and using the identity $\lambda_{pq}(e')\dfrac{n(f')}{n(f)} = \dfrac{n(e')}{n(f)}$ derived from Eq. (3.4), Eq. (3.6) becomes

$$(3.7) \qquad \left[\alpha(e') - \lambda_{pq}(e')\alpha(f')\right] - \left[\frac{n(e')}{n(e)}\alpha(e) - \frac{n(e')}{n(f)}\alpha(f)\right].$$

Clearly the first term in (3.7) is a multiple of $\alpha^\sharp(z_e^p z_f^q) = \alpha(pq)$. The second term is also a multiple of $\alpha(pq)$ by virtue of the condition $\dfrac{n(e)}{n(f)} = \lambda_{pq}(e)$.

The remaining cases are straight forward and their verification is left to the reader. Thus $\alpha^\sharp$ is a generalized axial function for the pre 1-skeleton $(\Gamma^\sharp, \theta^\sharp, \lambda^\sharp)$.

Definition 3.6. *The generalized 1-skeleton* $(\Gamma^\sharp, \alpha^\sharp, \theta^\sharp, \lambda^\sharp)$ *is called the* blow-up *of* $(\Gamma, \alpha, \theta, \lambda)$ *along the subskeleton* $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$.

Remark. *The blow-up construction for 1-skeleta was introduced by Guillemin and Zara [4]. However the assumptions on the 1-skeleta made in [4] are a bit more restrictive than those we make here. In particular they assume that*

1. α *is 3-independent and*
2. *the compatibility constants along the normal edges* N_0 *are all equal to 1; i.e.* $\lambda_e(e') = 1$ *for* $e \in E_0$ *and* $e' \in N_0^{i(e)}$.

Note that if condition 2 holds then $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$ *is level and if condition 1 holds then the function* $\alpha^\sharp$ *is actually 2-independent.*

3.3. **Reduction.**

3.3.1. *2-faces.* For 3-independent non-cyclic 1-skeleta, every 2-slice is a convex polygon and, in [4], these polygons are used to define the edges of cross sections. Unfortunately without the 3-independence condition these polygons cannot be detected by intersection alone. Fortunately we can recover these "hidden" polygons using the connection and a polarization.

Definition 3.7. *A* k-face *of a (polarized) generalized 1-skeleton* $(\Gamma, \alpha, \theta, \lambda)$ *is a* k-valent subskeleton $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$ *with* $b_0(\Gamma_0, \alpha_0) = 1$.

For example the 2-faces of a 3-independent noncyclic 1-skeleton are exactly its 2-valent 2-slices.

Definition 3.8. *We say that a (polarized) generalized 1-skeleton* $(\Gamma, \alpha, \theta, \lambda)$ *has* enough k-faces *if for each vertex* $p \in V_\Gamma$ *and any subset of* k *edges* $\{e_1, \ldots, e_k\} \in E^p$, *there is a unique* k-face containing $\{e_1, \ldots, e_k\}$.

Note that the 1-skeleton of a simple polytope has enough k-faces.

Definition 3.9. *A 1-skeleton* $(\Gamma, \alpha, \theta) \subset \mathbb{R}^n$ *is called* reducible *if*

R1. it admits a polarization and
R2. it has enough 2-faces.

In the 3-independent case the notion of noncyclic and reducible coincide. We will see presently that the reduction construction from [4] still yields something meaningful for reducible 1-skeleta.

3.3.2. *Cross sections.* For what follows it will be useful to keep track of the compatibility system of a 1-skeleton, so we shall include this in the notation. Fix $(\Gamma, \alpha, \theta, \lambda)$ a d-valent reducible 1-skeleton in $\mathbb{R}^n$ with generic polarizing covector $\xi \in (\mathbb{R}^n)^*$. Fix a ξ-compatible Morse function $\phi \colon V_\Gamma \to \mathbb{R}$ and a ϕ-regular value $c \in \mathbb{R}$.

A 2-face Q is a loop, and as such comes with two distinct orientations:
$$\begin{cases} Q := \{p_0 \to p_1 \to \cdots \to p_N \to p_0\} \\ \bar{Q} := \{p_0 \to p_N \to \cdots \to p_1 \to p_0\}. \end{cases}$$
An *oriented* 2-face is a 2-face with a fixed orientation. Let $\mathcal{F}_2$ denote the set of oriented 2-faces of $(\Gamma, \alpha, \theta, \lambda)$. Each oriented 2-face Q comes with a ξ-maximum vertex and a ξ-minimum vertex:

$$M_\xi(Q) = \max_{v \in Q}(\phi(v)) \quad \text{and} \quad m_\xi(Q) = \min_{v \in Q}(\phi(v)).$$

Define the c-vertex set V_c to be the oriented edges of Γ at c-*level*: $V_c = \{pq \in E_\Gamma \mid \phi(p) < c < \phi(q)\}$. Define the c-oriented edge set E_c to be the oriented 2-faces of Γ at c-level: $E_c = \left\{Q \in \mathcal{F}_2 \mid m_\xi(Q) < c < M_\xi(Q)\right\}$. The condition $b_0(Q) = 1$ implies that exactly two oriented edges in Q lie at c-level. Moreover exactly one of these oriented edges is a directed edge (via ξ) compatible with the orientation on Q; this directed edge is the initial c-vertex of the oriented c-edge Q. Hence the set E_c consists of ordered pairs of c-vertices, and the pair (V_c, E_c) defines a $(d-1)$-valent graph denoted by Γ_c.

The normal path-connection maps are used to define a connection on Γ_c. In fact there are two natural connections on Γ_c corresponding to the two paths around Q joining the two "c-vertices".

Fix $Q \in E_c$ and suppose $i(Q) = pq$ and $t(Q) = vw$ as in Figure 6. Define the *upper path* from q to w

(3.8) $\gamma_Q^u \colon q = r_1 \to r_2 \to \ldots \to r_{k-1} \to r_k = w,$

meaning that $\phi(r_i) > c$ for $1 \le i \le k$. Analogously define the *lower path* from p to v

(3.9) $\gamma_Q^d \colon p = t_1 \to t_2 \to \ldots \to t_{m-1} \to t_m = v,$

meaning that $\phi(t_j) < c$ for $1 \le j \le m$.

Since $(\Gamma, \alpha, \theta, \lambda)$ has enough 2-faces, the oriented edges normal to Q, $e \in N_Q^q$, are in one-to-one correspondence with oriented c-edges distinct from Q, $R_e \in (E_c)_{pq} \setminus \{Q\}$. Thus the normal path-connection maps for the upper path (resp. lower path) define the *up connection* (resp. *down*

connection) maps for Γ_c:

$$(3.10) \qquad \begin{array}{ccc} (E_c)_{pq} \setminus \{Q\} & \xrightarrow{(\theta_c^u)_Q} & (E_c)_{vw} \setminus \{\bar{Q}\} \\ {\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\ N_0^q & \xrightarrow[K_{\gamma_Q^u}^\perp]{} & N_0^w. \end{array}$$

The down connection maps are defined analogously, replacing $K_{\gamma_Q^u}^\perp$ with $K_{\gamma_Q^d}^\perp : N_0^p \to N_0^v$. Clearly the maps $(\theta_c^u)_Q$ (resp. $\left(\theta_c^d\right)_Q$) define a connection θ_c^u (resp. θ_c^d) on Γ_c called the *up connection* (resp. *down connection*). See Figure 6.
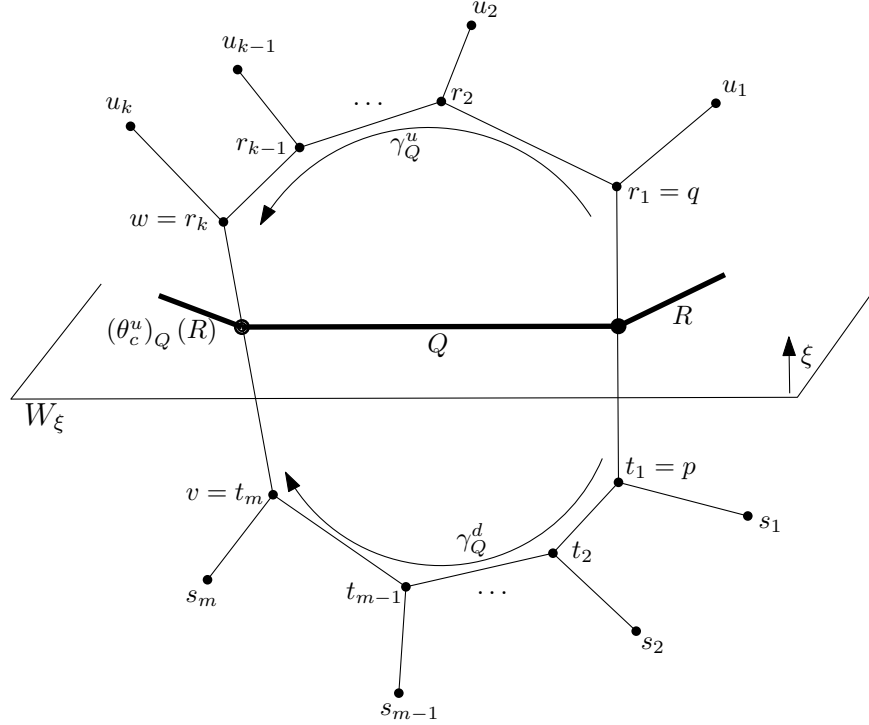

FIGURE 6. the c-cross-section

Just as we used path-connection maps to define the connection on Γ_c, we use the path-connection numbers to define a compatibility system on Γ_c.

Define the up compatibility maps by:

(3.11)

$$
\begin{array}{ccc}
(E_c)_{pq} \setminus \{Q\} & & \\
\cong \downarrow & \searrow^{(\lambda_c^u)_Q} & \\
& & \mathbb{R}_+ \\
N_0^q & \nearrow_{\left| K_{\gamma_Q^u}(-) \right|} &
\end{array}
$$

where the lower map is defined by $e \mapsto |K_{\gamma_Q^u}(e)|$ as in Definition 3.4. Define the down compatibility maps analogously replacing $\left| K_{\gamma_Q^u}(-) \right|$ by $\left| K_{\gamma_Q^d}(-) \right| : N_0^p \to \mathbb{R}^+$.

Observe that for every oriented c-edge $Q \in E_c$ and every $R_e \in (E_c)_{pq}$ we have

$$
(\lambda_c^u)_{\bar{Q}} \circ (\theta_c^u)_Q (R_e) = \left| K_{\bar{\gamma}_Q^u}(K_{\gamma_Q^u}(e)) \right| = \frac{1}{\left| K_{\gamma_Q^u}(e) \right|} = \frac{1}{(\lambda_c^u)_Q (R_e)},
$$

hence λ_c^u defines a compatiblity system for the pair (Γ_c, θ_c), and similarly for λ_c^d.

Therefore we have two (possibly distinct) pre-1-skeleta with the same underlying graph Γ_c, namely $(\Gamma_c, \theta_c^u, \lambda_c^u)$ and $(\Gamma_c, \theta_c^d, \lambda_c^d)$.

For each pre-1-skeleton defined above, we can define a compatible, generalized axial function on Γ_c as follows. Let $W_\xi \subset \mathbb{R}^n$ denote the subspace annihilated by ξ. Denote by $\bigwedge^2 \mathbb{R}^n$ the vector space of alternating two tensors generated by elements of the form $x \wedge y(= -y \wedge x)$ for $x, y \in \mathbb{R}^n$. Let $\iota: \bigwedge^2 \mathbb{R}^n \to \mathbb{R}^n$ denote the ξ-interior product map defined by $\iota(x \wedge y) = \langle \xi, x \rangle y - \langle \xi, y \rangle x$. As above let $Q \in E_c$ be an oriented c-edge with $i(Q) = pq$ and $t(Q) = vw$. Let

$$
\gamma_j^u: q = r_1 \to r_2 \to \ldots \to r_j
$$

be the partial upper path in Q from q to r_j and

$$
\gamma_j^d: p = t_1 \to t_2 \to \ldots \to t_j
$$

the partial lower path in Q from p to t_j. Then $\gamma_k^u = \gamma_Q^u: q \to \cdots \to r_k = w$ and $\gamma_m^d = \gamma_Q^d: p \to \cdots \to t_m = v$ as above. Our convention will be to let $p = r_0$ and $v = r_{k+1}$ and to let $q = t_0$ and $w = t_{m+1}$.

Define the function $\alpha_c^u: E_c \to W_\xi$ by

(3.12)
$$
\alpha_c^u(Q) = \frac{\iota(\alpha(r_1 r_0) \wedge \alpha(r_1 r_2))}{\langle \xi, \alpha(r_1 r_0) \rangle}.
$$

Similarly define the function $\alpha_c^d: E_c \to W_\xi$ by

(3.13)
$$
\alpha_c^d(Q) = \frac{\iota(\alpha(t_1 t_0) \wedge \alpha(t_1 t_2))}{\langle \xi, \alpha(t_1 t_0) \rangle}.
$$

An elegant argument due to Guillemin and Zara [4, Theorem 2.3.1] applies to show that the function α_c^u (resp. α_c^d) defines a generalized axial function for the pre 1-skeleton $(\Gamma_c, \theta_c^u, \lambda_c^u)$ (resp. $(\Gamma_c, \theta_c^d, \lambda_c^d)$). We refer the reader to [4] for the details.

Thus we get two (possibly distinct) generalized 1-skeleta structures on the $(d-1)$-valent graph Γ_c; the *up c-cross-section* of Γ, $(\Gamma_c, \alpha_c^u, \theta_c^u, \lambda_c^u)$, and the *down c-cross-section* of Γ, $(\Gamma_c, \alpha_c^d, \theta_c^d, \lambda_c^d)$.

An important component to the reduction technique is understanding what happens to cross sections as they pass over critical values. A beautiful description in the noncyclic case involving a blow-up construction was given in [4]; it turns out such a description is also valid in the reducible case.

3.3.3. *Passage over a critical value.* Let $(\Gamma, \alpha, \theta, \lambda)$ be a d-valent reducible 1-skeleton in $\mathbb{R}^n$. Fix a polarizing covector $\xi \in (\mathbb{R}^n)^*$ and a ξ-compatible Morse function ϕ. Fix two ϕ-regular values $c < c'$ such that there is a unique vertex $p \in V_\Gamma$ such that $c < \phi(p) < c'$.

Suppose that $\text{ind}_\xi(p) = r$ and let

$$V_{c,0} := \{p_i p \mid 1 \le i \le r\}$$

denote those edges flowing into p, i.e. $\langle \xi, \alpha(p_i p)\rangle > 0$ for $1 \le i \le r$. Let

$$V_{c',0} := \{p q_a \mid 1 \le a \le d - r\}$$

denote the oriented edges flowing out of p, i.e. $\langle \xi, \alpha(p q_a)\rangle > 0$ for $1 \le a \le d - r$.

Consider the up c-cross section $(\Gamma_c, \alpha_c^u, \theta_c^u, \lambda_c^u)$. The set of oriented edges $V_{c,0} \subset V_c$ is the c-vertex set of a totally geodesic complete subgraph $\Gamma_{c,0} \subset \Gamma_c$. Denote by Q_{ia} the oriented 2-face spanned by oriented edges $p p_i$ and $p q_a$ with initial c-vertex $p p_i$. The set of oriented c-edges normal to $\Gamma_{c,0}$ is denoted by $N_{c,0} := \{Q_{ia} \mid 1 \le a \le (d-r) \ 1 \le i \le r\}$. The function

$$(3.14) \qquad N_{c,0} \xrightarrow{\quad n^u \quad} \mathbb{R}_+$$

$$Q_{ia} \longmapsto \langle \xi, \alpha(p q_a)\rangle$$

defines a blow-up system for $(\Gamma_c^u, \alpha_c^u, \theta_c^u, \lambda_c^u)$ along $(\Gamma_{c,0}^u, \alpha_{c,0}^u, \theta_{c,0}^u, \lambda_{c,0}^u)$. Indeed for all i, j, and a we have

$$(3.15) \qquad \frac{n^u(Q_{ia})}{n^u\left((\theta_c^u)_{Q_{ij}}(Q_{ia})\right)} = \frac{n^u(Q_{ia})}{n^u(Q_{ja})} = 1 = (\lambda_c^u)_{Q_{ij}}(Q_{ia}).$$

We can therefore blow-up along the subskeleton to get the generalized 1-skeleton $(\Gamma_c^\sharp, (\alpha_c^u)^\sharp, (\theta_c^u)^\sharp, (\lambda_c^u)^\sharp)$.

Similarly we can blow-up the down c'-cross section $(\Gamma_{c'}, \alpha_{c'}^d, \theta_{c'}^d, \lambda_{c'}^d)$ along the $(d - r)$-valent totally geodesic subskeleton $(\Gamma_{c',0}, \alpha_{c',0}^d, \theta_{c',0}^d, \lambda_{c',0}^d)$. Here the c'-vertex set of $\Gamma_{c',0}$ is the oriented edge set $V_{c',0}$ and the oriented c'-edges normal to $\Gamma_{c',0}$ are the oriented 2-faces spanned by oriented edges pp_i and pq_a, denoted by Q_{ai} with $i(Q_{ai}) = pq_a$. In this case the function

$$(3.16) \qquad N_{c',0} \xrightarrow{ n^d } \mathbb{R}_+$$

$$Q_{ai} \longmapsto \langle \xi, \alpha(p_i p) \rangle$$

forms a blow-up system for $(\Gamma_{c'}, \alpha_{c'}^d, \theta_{c'}^d, \lambda_{c'}^d)$ along the subskeleton $(\Gamma_{c',0}, \alpha_{c',0}^d, \theta_{c',0}^d, \lambda_{c',0}^d)$ for similar reasons as above: for all $a, b,$ and i we have

$$(3.17) \qquad \frac{n^d (Q_{ai})}{n^u \left(\left(\theta_{c'}^d \right)_{Q_{ab}} (Q_{ai}) \right)} = \frac{n^d (Q_{ai})}{n^d (Q_{bi})} = 1 = \left(\lambda_c^d \right)_{Q_{ab}} (Q_{ai}).$$

We can therefore blow-up along the subskeleton to get the generalized 1-skeleton $(\Gamma_{c'}^\sharp, (\alpha_{c'}^d)^\sharp, (\theta_{c'}^d)^\sharp, (\lambda_{c'}^d)^\sharp)$. See Figure 7.
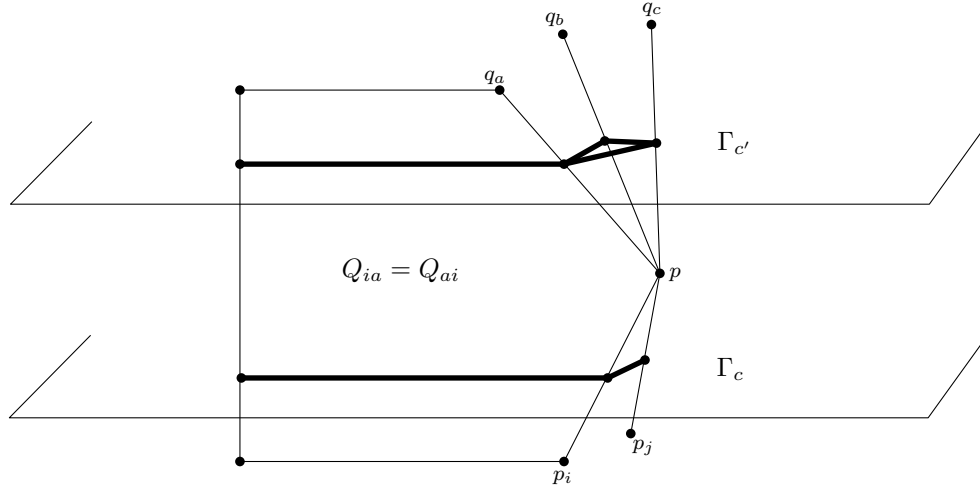

FIGURE 7. passage over a critical point

In [4] Guillemin and Zara prove that for 3-independent α, the pairs $(\Gamma_c^\sharp, (\alpha_c^u)^\sharp)$, $(\Gamma_{c'}^\sharp, (\alpha_{c'}^d)^\sharp)$ are "equivalent" in the sense that the graphs are isomorphic and the axial functions are positive multiples of one another. Unfortunately this statement fails to hold without further assumptions when one considers the whole generalized 1-skeleta quadruple including connections and compatibility systems. The following lemma is an analogue of Theorem 2.3.2 in [4].

Lemma 1. *Assume that all 2-faces at c or c' are level and have trivial normal holonomy. Then we have the following equivalences of 1-skeleta.*

(i) $(\Gamma_c, \alpha_c^u, \theta_c^u, \lambda_c^u) \equiv (\Gamma_c, \alpha_c^d, \theta_c^d, \lambda_c^d)$.

(ii) $(\Gamma_c^\sharp, (\alpha_c^u)^\sharp, (\theta_c^u)^\sharp, (\lambda_c^u)^\sharp) \equiv (\Gamma_{c'}^\sharp, (\alpha_{c'}^d)^\sharp, (\theta_{c'}^d)^\sharp, (\lambda_{c'}^d)^\sharp)$.

Proof. First we establish (i). Fix oriented 2-faces Q and R as shown in Figure 6, let γ_Q^u (resp. γ_Q^d) be as in Eq. (3.8) (resp. (3.9)). Identify the oriented c-edges at $i(Q) = pq$ with the oriented edges normal to Q at p (resp. q), i.e. $(E_c)^{pq} \setminus \{Q\} \cong N_Q^p \left(\text{resp.} N_Q^q\right)$. Similarly identify the oriented c-edges at $t(Q) = vw$ with the oriented edges normal to Q at v (resp. w), i.e. $(E_c)_{vw} \setminus \{\bar{Q}\} \cong N_Q^v \left(\text{resp.} N_Q^w\right)$. We have the following diagram:

$$
\begin{array}{ccc}
(E_c)_{\overline{pq}} \setminus \{Q\} & \xrightarrow{(\theta_c^u)_Q} & (E_c)_{vw} \setminus \{\bar{Q}\} \\
{\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\
N_Q^q & \xrightarrow{K_{\gamma_Q^u}^\perp} & N_Q^w \\
{\scriptstyle\theta_{qp}^\perp}\big\downarrow & & \big\downarrow{\scriptstyle\theta_{vw}^\perp} \\
N_Q^p & \xrightarrow{K_{\gamma_Q^d}^\perp} & N_Q^v \\
{\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\
(E_c)_{pq} \setminus \{Q\} & \xrightarrow{(\theta_c^d)_Q} & (E_c)_{vw} \setminus \{\bar{Q}\}.
\end{array}
$$

The vertical maps compose to the identity maps on $(E_c)_{pq} \setminus \{Q\}$ and $(E_c)_{vw} \setminus \{Q\}$. The top and bottom squares commute by definition, i.e. Eq. (3.10), and the commutativity of the middle square follows from the trivial normal holonomy of Q. Thus the outer rectangle commutes, which implies the equivalence of θ_c^u and θ_c^d.

Using Eqs. (3.12) and (3.13), and that

$$\alpha(qr_2) - \lambda_{qp}(qr_2)\alpha(pt_2) = c\alpha(qp) = -c\alpha(pq),$$

we see $\alpha_c^u(Q) = \kappa(Q) \cdot \alpha_{c'}^d(Q)$ where

(3.18) $$\kappa(Q) := \lambda_{qp}(qr_2).$$

Moreover, the trivial normal holonomy of Q implies that

$$K_{\gamma_Q^d}^\perp \circ \theta_{qp}^\perp = \theta_{wv}^\perp \circ K_{\gamma_Q^u}^\perp.$$

Let $R \in (E_c)_{pq} \setminus \{Q\}$ be an oriented c-edge at $i(Q) = pq$ corresponding to oriented edges $qu_1 \in N_q^0$ and $ps_1 \in N_p^0$. Then the levelness of Q implies

$$\left| K_{\gamma_Q^d}^\perp(ps_1) \right| \cdot \lambda_{qp}(qu_1) = \lambda_{wv}\left(K_{\gamma_Q^u}^\perp(qu_1) \right) \cdot \left| K_{\gamma_Q^u}^\perp(qu_1) \right|,$$

from which it follows that

$$\left(\lambda_c^d \right)_Q (R) \cdot \frac{\kappa(R)}{\kappa\left((\theta_c^u)_Q (R) \right)} = (\lambda_c^u)_Q (R).$$

Now we establish (ii). There is a natural identification of the vertices which extends to an identification of graphs $\Gamma_c^\sharp$ and $\Gamma_{c'}^\sharp$:

$$(3.19) \qquad\qquad V_c \setminus V_{c,0} =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= V_{c'} \setminus V_{c',0} \ .$$

$$\sqcup \qquad\qquad\qquad\qquad \sqcup$$

$$\{z_{ia}\} \xrightarrow{\hspace{3cm}} \{z_{ai}\}$$

Note that c-vertices outside the singular locus of $\Gamma_c^\sharp$ coincide with the c'-vertices outside the singular locus of $\Gamma_{c'}^\sharp$. Indeed oriented edges not containing p are at c-level if and only if they are at c'-level. Also note that Eq. (3.19) identifies horizontal edges of $\Gamma_{c,0}^\sharp$ with vertical edges of $\Gamma_{c',0}^\sharp$ and vice versa. In order to establish (ii), we need to show

(a) $(\theta_c^u)_\epsilon^\sharp = \left(\theta_{c'}^d \right)_\epsilon^\sharp$

(b) $(\lambda_c^u)_\epsilon^\sharp = \dfrac{\kappa}{\kappa \circ (\theta_c^u)_\epsilon^\sharp} \left(\lambda_{c'}^d \right)_\epsilon^\sharp$

(c) $(\alpha_c^u)^\sharp = \kappa\left(\alpha_{c'}^d \right)^\sharp.$

Note that for oriented edges not issuing from the singular locus of either $\Gamma_c^\sharp$ or $\Gamma_{c'}^\sharp$, the blow-up connection maps, compatibility systems, and generalized axial functions coincide with their counter parts on Γ_c or $\Gamma_{c'}$, cf. Eqs. (3.1), (3.2), and (3.3). Therefore by (i), it suffices to establish (a), (b), and (c) for oriented edges ϵ issuing from the singular locus of $\Gamma_c^\sharp \cong \Gamma_{c'}^\sharp$.

A direct comparison of values in Table 4 shows that $(\theta_c^u)_\epsilon^\sharp = \left(\theta_{c'}^d \right)_\epsilon^\sharp$, hence (a) holds.

Tables 5 and 6 compare the compatibility systems and generalized axial functions on $\Gamma_c^\sharp$ and $\Gamma_{c'}^\sharp$.

ϵ' \ ϵ	$Z_{ia}[t(Q_{ia})]$	$Z_{ia}Z_{ja}$	$Z_{ia}Z_{ib}$
$Z_{ia}[t(Q_{ia})]$	-	$Z_{ja}[t(Q_{ja})]$	$Z_{ib}[t(Q_{ib})]$
$Z_{ia}Z_{ka}$	$(\theta^u_c)_{Q_{ia}}(Q_{ik})$	$Z_{ja}Z_{ka}$	$Z_{ib}Z_{kb}$
$Z_{ia}Z_{ic}$	$(\theta^u_c)_{Q_{ia}}(Q_{ic})$	$Z_{ja}Z_{jc}$	$Z_{ib}Z_{ic}$
$Z_{ai}[t(Q_{ai})]$	-	$Z_{aj}[t(Q_{aj})]$	$Z_{bi}[t(Q_{bi})]$
$Z_{ai}Z_{ak}$	$\left(\theta^d_{c'}\right)_{Q_{ai}}(Q_{ak})$	$Z_{aj}Z_{ak}$	$Z_{bi}Z_{kb}$
$Z_{ai}Z_{ci}$	$\left(\theta^d_{c'}\right)_{Q_{ai}}(Q_{ci})$	$Z_{aj}Z_{cj}$	$Z_{bi}Z_{ci}$
ϵ' \ ϵ	$Z_{ai}[t(Q_{ai})]$	$Z_{ai}Z_{aj}$	$Z_{ai}Z_{bi}$

TABLE 4. $(\theta^u_c)^\sharp_\epsilon(\epsilon') = \left(\theta^d_{c'}\right)^\sharp_\epsilon(\epsilon')$

ϵ' \ ϵ	$Z_{ia}[t(Q_{ia})]$	$Z_{ia}Z_{ja}$	$Z_{ia}Z_{ib}$
$Z_{ia}[t(Q_{ia})]$	-	1	1
$Z_{ia}Z_{ka}$	$(\lambda^u_c)_{Q_{ia}}(Q_{ik})$	$(\lambda^u_c)_{Q_{ij}}(Q_{ik})$	1
$Z_{ia}Z_{ic}$	$(\lambda^u_c)_{Q_{ia}}(Q_{ic})$	$(\lambda^u_c)_{Q_{ij}}(Q_{ic})$	1
$Z_{ai}[t(Q_{ai})]$	-	1	1
$Z_{ai}Z_{ak}$	$\left(\lambda^d_{c'}\right)_{Q_{ai}}(Q_{ak})$	1	$\left(\lambda^d_{c'}\right)_{Q_{ab}}(Q_{ak})$
$Z_{ai}Z_{ci}$	$\left(\lambda^d_{c'}\right)_{Q_{ai}}(Q_{ci})$	1	$\left(\lambda^d_{c'}\right)_{Q_{ab}}(Q_{ac})$
ϵ' \ ϵ	$Z_{ai}[t(Q_{ai})]$	$Z_{ai}Z_{aj}$	$Z_{ai}Z_{bi}$

TABLE 5. $(\lambda^u_c)^\sharp_\epsilon(\epsilon') \equiv \left(\lambda^d_{c'}\right)^\sharp_\epsilon(\epsilon')$

ϵ	$(\alpha^u_c)^\sharp(\epsilon)$	$\left(\alpha^d_{c'}\right)^\sharp(\epsilon)$	ϵ
$Z_{ia}[t(Q_{ia})]$	$\frac{1}{n^u(Q_{ia})}\alpha^u_c(Q_{ia})$	$\frac{1}{n^d(Q_{ai})}\alpha^u_c(Q_{ai})$	$Z_{ai}[t(Q_{ai})]$
$Z_{ia}Z_{ja}$	$\alpha^u_c\left(Q_{ij}\right)$	$\alpha^d_{c'}\left(Q_{aj}\right) - \dfrac{n^d\left(Q_{aj}\right)}{n^d\left(Q_{ai}\right)}\alpha^d_{c'}\left(Q_{ai}\right)$	$Z_{ai}Z_{aj}$
$Z_{ia}Z_{ib}$	$\alpha\left(Q_{ib}\right) - \dfrac{n\left(Q_{ib}\right)}{n\left(Q_{ia}\right)}\alpha\left(Q_{ia}\right)$	$\alpha^d_{c'}\left(Q_{ab}\right)$	$Z_{ai}Z_{bi}$

TABLE 6. $(\alpha^u_c)^\sharp \equiv \left(\alpha^d_{c'}\right)^\sharp$

Recalling that
$$
\begin{cases}
\alpha_c^u\left(Q_{ij}\right) = & \alpha\left(pp_j\right) - \frac{\langle \alpha(pp_j),\xi \rangle}{\langle \alpha(pp_i),\xi \rangle}\alpha\left(pp_i\right)\\[2ex]
\alpha_c^u\left(Q_{ia}\right) = & \alpha\left(pq_a\right) - \frac{\langle \alpha(pq_a),\xi \rangle}{\langle \alpha(pp_i),\xi \rangle}\alpha\left(pp_i\right)\\[2ex]
\alpha_{c'}^d\left(Q_{ai}\right) = & \alpha\left(pp_i\right) - \frac{\langle \alpha(pp_i),\xi \rangle}{\langle \alpha(pq_a),\xi \rangle}\alpha\left(pq_a\right)\\[2ex]
\alpha_{c'}^d\left(Q_{ab}\right) = & \alpha\left(pq_b\right) - \frac{\langle \alpha(pq_b),\xi \rangle}{\langle \alpha(pq_a),\xi \rangle}\alpha\left(pq_a\right)
\end{cases}
$$

and that
$$
\begin{cases}
n^u\left(Q_{ia}\right) = & \langle \alpha\left(pq_a\right),\xi \rangle\\
n^d\left(Q_{ai}\right) = & \langle \alpha\left(p_i p\right),\xi \rangle,
\end{cases}
$$

it is straight forward to check that

$$
\begin{aligned}
\tfrac{1}{n^u(Q_{ia})}\alpha_c^u\left(Q_{ia}\right) &= & 1 & \cdot\left(\tfrac{1}{n^d(Q_{ai})}\alpha_{c'}^d\left(Q_{ai}\right)\right)\\
\alpha_c^u\left(Q_{ij}\right) &= & \tfrac{1}{\langle \alpha(p_i p),\xi \rangle} & \cdot\left(\alpha_{c'}^d\left(Q_{aj}\right) - \tfrac{n^d(Q_{aj})}{n^d(Q_{ai})}\alpha_{c'}^d\left(Q_{ai}\right)\right)\\
\alpha_c^u\left(Q_{ib}\right) - \tfrac{n^u(Q_{ib})}{n^u(Q_{ia})}\alpha_c^u\left(Q_{ia}\right) &= & \langle \alpha(pq_a),\xi \rangle & \cdot\left(\alpha_{c'}^d\left(Q_{ab}\right)\right).
\end{aligned}
$$

Thus we deduce that the values of $\kappa(\epsilon)$ that make (c) hold are as shown in Table 7. The verification that (b) also holds with κ defined by Table 7 is straight forward, and is left to the reader. This establishes the equivalence in (ii), and thereby completes the proof of Lemma 1.

ϵ	$\kappa(\epsilon)$
Q	$\lambda_{qp}(qr_2)$
$Z_{ia}[t(Q_{ia})]$	1
$Z_{ia}Z_{ja}$	$\frac{1}{\langle \alpha(p_i p),\xi \rangle}$
$Z_{ia}Z_{ib}$	$\langle \alpha(pq_a),\xi \rangle$

TABLE 7. Values of κ

□

3.4. **Cutting.** Let $(\Gamma, \alpha, \theta, \lambda)$ be a d-valent reducible 1-skeleton in $\mathbb{R}^n$, ξ a polarizing covector in $(\mathbb{R}^n)^*$, and ϕ a compatible Morse function. Let $I = (\{0, 1\}, \{01, 10\})$ denote the interval graph (i.e. connected single edge graph), and let θ_I denote the unique connection on I. Then $\lambda_I \equiv 1$ defines a compatibility system on (I, θ_I) and the function $\alpha_I\colon \begin{cases} 01 \mapsto 1\\ 10 \mapsto -1 \end{cases}$ defines an axial function, making the quadruple $(I, \alpha_I, \theta_I, \lambda_I)$ a 1-skeleton in $\mathbb{R}$.

The *direct product* 1-skeleton $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda}) \subset \mathbb{R}^n \times \mathbb{R}$ with factors $(\Gamma, \alpha, \theta, \lambda) \subset \mathbb{R}^n$ and $(I, \alpha_I, \theta_I, \lambda_I) \subset \mathbb{R}$ is a $(d+1)$-valent 1-skeleton defined as follows. The graph $\hat{\Gamma}$ has vertex set $V_{\hat{\Gamma}} := V_\Gamma \times 0 \sqcup V_\Gamma \times 1$ and oriented edge set

$E_{\hat{\Gamma}} := E_\Gamma \times V_I \sqcup V_\Gamma \times E_I$. The connection maps are defined as the product of the connection maps of the factors: $\hat{\theta}_\epsilon := (\theta \times \theta_I)_\epsilon : E_{\hat{\Gamma}}^{i(\epsilon)} \to E_{\hat{\Gamma}}^{t(\epsilon)}$. The compatibility system maps are likewise the product of the compatibility system maps of the factors: $\hat{\lambda}_\epsilon := (\lambda \times \lambda_I)_\epsilon : E_{\hat{\Gamma}}^{i(\epsilon)} \to \mathbb{R}_+$. The product axial function

is $E_{\hat{\Gamma}} \xrightarrow{\quad \hat{\alpha} \quad} \mathbb{R}^n \times \mathbb{R}$

$e \times v \longmapsto (\alpha(e), 0)$

$v \times e \longmapsto (0, 1)$.

Let $\mathbf{1} \in \mathbb{R}^*$ denote the linear function on $\mathbb{R}$ that maps 1 to 1. Define the covector $\hat{\xi} := (\xi, \mathbf{1}) \in (\mathbb{R}^n \times \mathbb{R})^* \cong (\mathbb{R}^n)^* \times \mathbb{R}^*$. Then $\hat{\xi}$ is generic and polarizing for $(\hat{\Gamma}, \hat{\alpha}_\eta, \hat{\theta}, \hat{\lambda})$ since

$$\langle \hat{\xi}, \hat{\alpha}(e \times v) \rangle = \langle \xi, \alpha(e) \rangle$$
$$\langle \hat{\xi}, \hat{\alpha}(v \times e) \rangle = \langle \mathbf{1}, \alpha_I(e) \rangle.$$

Also note that $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$ has enough 2-faces; 2-faces are either $Q \times v$ for Q a 2-face of $(\Gamma, \alpha, \theta, \lambda)$ or rectangles $e \times e'$. Thus $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is reducible.

Set $\phi_- := \min_{p \in V_\Gamma}(\phi(p))$ and $\phi_+ := \max_{p \in V_\Gamma}(\phi(p))$, and fix $a > \phi_+ - \phi_- > 0$. Define a $\hat{\xi}$-compatible Morse function by

$$V_{\hat{\Gamma}} \xrightarrow{\quad \hat{\phi} \quad} \mathbb{R}$$

$$v \times t \longmapsto \phi(v) + at.$$

Lemma 2. *In the notation above, if $(\Gamma, \alpha, \theta, \lambda)$ satisfies (†) in Theorem 1, then the direct product 1-skeleton $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$ also satisfies (†). Moreover if $c \in \mathbb{R}$ is any $\hat{\phi}$-regular value such that*

$$\phi_+ < c < \phi_- + a$$

then there is a linear isomorphism $\pi \colon W_{\hat{\xi}} \to \mathbb{R}^n$ that yields an equivalence of 1-skeleta $(\hat{\Gamma}_c, \pi \circ \hat{\alpha}_c^d, \hat{\theta}_c^d, \hat{\lambda}_c^d) \equiv (\Gamma, \alpha, \theta, \lambda)$.

Proof. Using the notation above, assume that $(\Gamma, \alpha, \theta, \lambda)$ satisfies (†). The direct product has two types of 2-faces:

    (i) 2-faces of the form $Q \times v$, for Q a 2-face of $(\Gamma, \alpha, \theta, \lambda)$.

    (ii) Rectangles of the form $e \times e'$, for $e \in E_\Gamma$ and $e' \in E_I$.

Clearly 2-faces of type (i) are level with trivial normal holonomy since the $\hat{\theta}$ agrees with θ and $\hat{\lambda}$ agrees with λ on $Q \times v$. Let $F := pq \times 01$ be a 2-face of type (ii), and let γ_F denote the loop $p \times 0 \to q \times 0 \to q \times 1 \to p \times 1 \to p \times 0$. Then the normal holonomy map $K_{\gamma_F}^\perp$ has the form

$$K_{\gamma_F}^\perp = (\theta_I)_{10} \circ \theta_{qp} \circ (\theta_I)_{01} \circ \theta_{pq},$$

which is clearly the identity map on $N_F^{p\times 0}$. Moreover the normal holonomy number has the form

$$\left|K_{\gamma_F}^{\perp}\right| = (\lambda_I)_{10} \cdot \lambda_{qp} \cdot (\lambda_I)_{01} \cdot \lambda_{pq},$$

which is identically one on $N_F^{p\times 0}$. This shows that (†) holds for $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$.

To establish the equivalence of $(\Gamma, \alpha, \theta, \lambda)$ with the down c-cross section of $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$, first note that the oriented edges at c-level are those of the form $v\times 01$. Indeed all vertices of the form $p\times 0$ have $\hat{\phi}(p \times 0) = \phi(p) \le \phi_+ < c$ and all vertices of the form $q \times 1$ have $\hat{\phi}(q \times 1) = \phi(q) + a \ge \phi_- + a > c$. Similarly the oriented 2-faces at c-level are the oriented rectangles of the form $e \times 01$. This defines a bijection between graphs Γ and $\hat{\Gamma}_c$:

$$V_\Gamma \sqcup E_\Gamma \longrightarrow \hat{V}_c \sqcup \hat{E}_c$$

$$v \longmapsto v \times 01$$

$$e \longmapsto e \times 01$$

.

The down connection map $\left(\hat{\theta}_c^d\right)_{e\times 01} = K_{\gamma^e}^{\perp}$ is simply equal to θ_e by definition. Likewise, the down compatibility map $\left(\hat{\lambda}_c^d\right)_{e\times 01} = \left|K_{\gamma^e}^{\perp}\right|$ is equal to λ_e. Finally we have

$$\hat{\alpha}_c^d (pq \times 01) = \hat{\alpha}(pq \times 0) - \frac{\langle \hat{\xi}, \hat{\alpha}(pq \times 0)\rangle}{\langle \hat{\xi}, \hat{\alpha}(p \times 01)\rangle}\hat{\alpha}(p \times 01)$$

$$= \begin{pmatrix} \alpha(pq) \\ 0 \end{pmatrix} - \frac{\langle \xi, \alpha(pq)\rangle}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(pq) \\ -\langle \xi, \alpha(pq)\rangle \end{pmatrix}.$$

Then the restriction of the projection map $\pi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ to the vanishing hyperplane of $\hat{\xi}$, $W_{\hat{\xi}}$, is a linear isomorphism such that $\pi \circ \hat{\alpha}_c^d = \alpha$. □

4. The Main Result

4.1. **Projections of 1-skeleta.** Fix a d-valent pre 1-skeleton $(\Gamma, \theta, \lambda)$, let $A \colon E_\Gamma \to \mathbb{R}^N$ be an effective generalized axial function and let $p \colon \mathbb{R}^N \to \mathbb{R}^n$ be any surjective linear map. Then $p \circ A$, is also an effective generalized
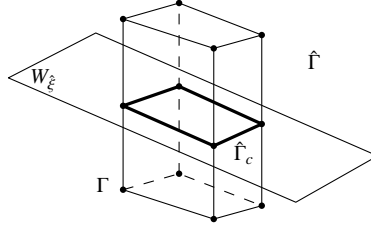
FIGURE 8. cutting

axial function for $(\Gamma, \theta, \lambda)$, and the resulting generalized 1-skeleton, $(\Gamma, p \circ A, \theta, \lambda)$, is called the *projection* of $(\Gamma, A, \theta, \lambda)$.

Conversely, we say a generalized 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ in $\mathbb{R}^n$ has a *lift* if there is an effective generalized 1-skeleton $(\Gamma, A, \theta, \lambda)$ in $\mathbb{R}^N$ ($n < N \le d$) and a surjective linear map $p \colon \mathbb{R}^N \to \mathbb{R}^n$ such that $\alpha = p \circ A$. We say $(\Gamma, \alpha, \theta, \lambda)$ has a *total lift* if it has a lift to an effective generalized 1-skeleton $(\Gamma, A, \theta, \lambda)$ in $\mathbb{R}^d$. Note that an effective axial function $A \colon E_\Gamma \to \mathbb{R}^d$ for a d-valent pre 1-skeleton $(\Gamma, \theta, \lambda)$ is necessarily d-independent.
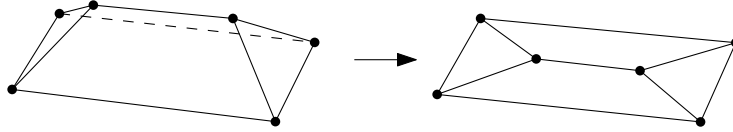

FIGURE 9. Projection

For the convenience of the reader, we restate Theorem 1 here.

Theorem 1. *Let* $(\Gamma, \alpha, \theta, \lambda) \subset \mathbb{R}^n$ *be a d-valent reducible 1-skeleton in* $\mathbb{R}^n$. *Then* $(\Gamma, \alpha, \theta, \lambda)$ *has a total lift if and only if*

(†) *Every 2-face of* $(\Gamma, \alpha, \theta, \lambda)$ *is level and has trivial normal holonomy.*

Note that a (total) lift $(\Gamma, A, \theta, \lambda)$ of a reducible 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ is necessarily reducible. Indeed any generic covector $\xi \in (\mathbb{R}^n)^*$ for $(\Gamma, \alpha, \theta, \lambda)$ pulls back via the projection $p \colon \mathbb{R}^d \to \mathbb{R}^n$ to a generic covector $\Xi \in (\mathbb{R}^d)^*$ for $(\Gamma, A, \theta, \lambda)$ satisfying $\langle \Xi, A(e) \rangle = \langle \xi, \alpha(e) \rangle$. Hence the 2-faces of $(\Gamma, A, \theta, \lambda)$ and $(\Gamma, \alpha, \theta, \lambda)$ have the same underlying totally geodesic subgraphs.

4.2. **Proof of Theorem 1.** Before proving Theorem 1 we will need the following lemmata. The first lemma asserts that total liftability is preserved by a blow-up or a blow-down.

Fix a d-valent generalized 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ in $\mathbb{R}^n$, a k-valent (level) subskeleton $(\Gamma_0, \alpha_0, \theta_0, \lambda_0)$, and a blow-up system for the subskeleton $n \colon N_0 \to \mathbb{R}_+$. Let $(\Gamma^\sharp, \alpha^\sharp, \theta^\sharp, \lambda^\sharp)$ denote the corresponding blow-up generalized 1-skeleton.

Lemma 3. $(\Gamma, \alpha, \theta, \lambda)$ *has a total lift if and only if* $(\Gamma^\sharp, \alpha^\sharp, \theta^\sharp, \lambda^\sharp)$ *has a total lift.*

Proof. Let $(\Gamma, A, \theta, \lambda)$ be a total lift of $(\Gamma, \alpha, \theta, \lambda)$ via a surjective map $p \colon \mathbb{R}^d \to \mathbb{R}^n$. Then the function $A^\sharp \colon E^\sharp \to \mathbb{R}^d$ defined as in Definition 3.6 is an axial function for $(\Gamma^\sharp, \theta^\sharp, \lambda^\sharp)$. Moreover the linearity of p guarantees that $p \circ A^\sharp = \alpha^\sharp$. Finally note that $\left\{ A^\sharp(e) \mid e \in E^p \right\}$ and $\{A(e) \mid e \in E^p\}$ coincide for $p \notin V_0^\sharp$. Thus since A is d-independent, $A^\sharp$ must also be d-independent.

Conversely let $(\Gamma^\sharp, \tilde{A}, \theta^\sharp, \lambda^\sharp)$ be a total lift of $(\Gamma^\sharp, \alpha^\sharp, \theta^\sharp, \lambda^\sharp)$ via a surjective map $p \colon \mathbb{R}^d \to \mathbb{R}^n$. In order to define a total lift $A \colon E_\Gamma \to \mathbb{R}^d$ for α, it suffices to see that $\tilde{A}$ is constant on the fibers of the blow-up morphism. Indeed if A is constant on the fibers $\beta^{-1}(e)$ for each $e \in E_0$, then the function $A \colon E_\Gamma \to \mathbb{R}^d$

$$(4.1) \qquad A(e) = \begin{cases} n(e)\tilde{A}(z_e^p[t(e)]) & \text{if } e \in N_0 \\ \tilde{A}(\epsilon) & \text{if } e \in E_\Gamma \setminus N_0 \text{ and } \epsilon \in \beta^{-1}(e) \end{cases}$$

is well-defined. Fix a vertex $p \in V_\Gamma$, oriented edges $e, e' \in E^p$, and consider the difference

$$(4.2) \qquad A(e') - \lambda_e(e')A(\theta_e(e')).$$

Fix $\epsilon \in \beta^{-1}(e)$ and $\epsilon' \in \beta^{-1}(e')$. For $e' \in N_0$, Eq. (4.2) becomes

$$n(e')\tilde{A}(\epsilon') - \lambda_e(e') \cdot n(\theta_e(e'))\tilde{A}(\theta_\epsilon^\sharp(\epsilon')) = n(e')\left(\tilde{A}(\epsilon') - 1 \cdot \tilde{A}(\theta_\epsilon^\sharp(\epsilon'))\right),$$

which is a multiple of $\tilde{A}(\epsilon) = \frac{1}{n(e)}A(e)$. For $e' \notin N_0$, Eq. (4.2) becomes

$$\tilde{A}(\epsilon') - \lambda_\epsilon^\sharp(\epsilon')\tilde{A}(\theta_\epsilon^\sharp(\epsilon')),$$

which is also a multiple of $\tilde{A}(\epsilon) = A(e)$. Thus Eq. (4.1) defines a generalized axial function for $(\Gamma, \theta, \lambda)$. Moreover it is clear that A is d-independent (since $\tilde{A}$ is d-independent), and that $p \circ A = \alpha$. Thus it remains to see that $\tilde{A}$ is constant on $\beta^{-1}(e)$ for each $e \in E_0$.

Fix $e \in E_0$ and let ϵ', ϵ'' be any two edges in the fiber $\beta^{-1}(e)$. We need to show that $\tilde{A}(\epsilon') = \tilde{A}(\epsilon'')$. If ϵ' and ϵ'' are not distinct there is nothing to show. Otherwise ϵ' and ϵ'' are distinct edges in the same fiber, hence there must be a vertical edge ϵ joining the vertices $i(\epsilon')$ and $i(\epsilon'')$. Thus we have

$$(4.3) \qquad \tilde{A}(\epsilon') - \lambda_\epsilon^\sharp(\epsilon')\tilde{A}(\epsilon'') = c\tilde{A}(\epsilon)$$

for some $c \in \mathbb{R}$. The claim is that $c = 0$. To see this, apply the projection p to both sides of (4.3) to get

$$(4.4) \qquad \alpha^\sharp(\epsilon') - \lambda_\epsilon^\sharp(\epsilon')\alpha^\sharp(\epsilon'') = c\alpha^\sharp(\epsilon);$$

Since $\lambda_\epsilon^\sharp(\epsilon') = 1$ and $\alpha^\sharp(\epsilon') = \alpha^\sharp(\epsilon'')$, we deduce that the LHS of (4.4) must be zero. This implies that c must also be equal to zero, as desired. $\square$

The next lemma asserts that any down cross section of a reducible 1-skeleton satisfying (†) must have a total lift. See Figure 10.

Lemma 4. *Let $(\Gamma, \alpha, \theta, \lambda)$ be a d-valent reducible 1-skeleton satisfying (†) in Theorem 1, ξ a polarizing covector for $(\Gamma, \alpha, \theta, \lambda)$, and ϕ a compatible Morse function. Then for any ϕ-regular value $c \in \mathbb{R}$, the $(d-1)$-valent generalized 1-skeleton $(\Gamma_c, \alpha_c^d, \theta_c^d, \lambda_c^d)$ has a total lift.*

Proof. Let $c_1 < \cdots < c_N$ be ϕ-regular values such that for each $1 \le i \le N-1$ there is a unique vertex $p_i \in V_\Gamma$ such that $c_i < \phi(p_i) < c_{i+1}$. Since $(\Gamma, \alpha, \theta, \lambda)$ is reducible it has zeroth combinatorial Betti number equal to 1. Hence there are unique vertices p_0 and p_N such that $\phi(p_0) < c_1 < c_N < \phi(p_N)$. To complete the proof it suffices to show that $(\Gamma_{c_1}, \alpha_{c_1}^d, \theta_{c_1}^d, \lambda_{c_1}^d)$ has a total lift. Indeed inductively assume that $(\Gamma_{c_i}, \alpha_{c_i}^d, \theta_{c_i}^d, \lambda_{c_i}^d)$ has a total lift. Since all 2-faces are level with trivial normal holonomy, Lemma 1 implies that $(\Gamma_{c_i}, \alpha_{c_i}^u, \theta_{c_i}^u, \lambda_{c_i}^u) \equiv (\Gamma_{c_i}, \alpha_{c_i}^d, \theta_{c_i}^d, \lambda_{c_i}^d)$. Hence $(\Gamma_{c_i}, \alpha_{c_i}^u, \theta_{c_i}^u, \lambda_{c_i}^u)$ must also have a total lift. By Lemma 3, the blow-up $(\Gamma_{c_i}^\sharp, (\alpha_{c_i}^u)^\sharp, (\theta_{c_i}^u)^\sharp, (\lambda_{c_i}^u)^\sharp)$ has a total lift. Again Lemma 1 implies that $(\Gamma_{c_i}^\sharp, (\alpha_{c_i}^u)^\sharp, (\theta_{c_i}^u)^\sharp, (\lambda_{c_i}^u)^\sharp) \equiv (\Gamma_{c_{i+1}}^\sharp, (\alpha_{c_{i+1}}^d)^\sharp, (\theta_{c_{i+1}}^d)^\sharp, (\lambda_{c_{i+1}}^d)^\sharp)$. Hence $(\Gamma_{c_{i+1}}^\sharp, (\alpha_{c_{i+1}}^d)^\sharp, (\theta_{c_{i+1}}^d)^\sharp, (\lambda_{c_{i+1}}^d)^\sharp)$ has a total lift as well. Hence by Lemma 3 again, $(\Gamma_{c_{i+1}}, \alpha_{c_{i+1}}^d, \theta_{c_{i+1}}^d, \lambda_{c_{i+1}}^d)$ must also have a lift, and so on.

In order to show that the c_1-cross section $(\Gamma_{c_1}, \alpha_{c_1}^d, \theta_{c_1}^d, \lambda_{c_1}^d)$ has a lift, note that the graph Γ_{c_1} is a complete graph on d vertices. For concreteness, set $p := p_0$, $E^p = \{pq_i \mid 1 \le i \le d\} = V_{c_1}$ the c_1-vertices, and $E_{c_1} := \{Q_{ij}\}$ the oriented c_1-edges (where Q_{ij} is the oriented 2-face spanned by the oriented edges $i(Q_{ij}) = pq_i$ and $t(Q_{ij}) = pq_j$). Note that the down connection on Γ_{c_1} gives $(\theta_c^d)_{Q_{ij}}(Q_{ik}) = Q_{jk}$ for all $1 \le i, j, k \le d$. Also by definition the compatibility system gives $\left(\lambda_{c_1}^d\right)_{Q_{ij}}(Q_{ik}) = 1$ for all $1 \le i, j, k \le d$. Define constants $\left\{ m_{ij} := \dfrac{\langle \xi, \alpha(pq_i) \rangle}{\langle \xi, \alpha(pq_j) \rangle} \mid 1 \le i, j \le d \right\}$. For all $1 \le i, j, k \le d$ we have

$$(4.5) \qquad \alpha_{c_1}^d \left(Q_{ij} \right) = -m_{ji} \cdot \alpha_{c_1}^d \left(Q_{ji} \right)$$

$$(4.6) \qquad \alpha_{c_1}^d \left(Q_{ik} \right) - \alpha_{c_1}^d \left(Q_{jk} \right) = m_{kj} \cdot \alpha_{c_1}^d \left(Q_{ij} \right).$$
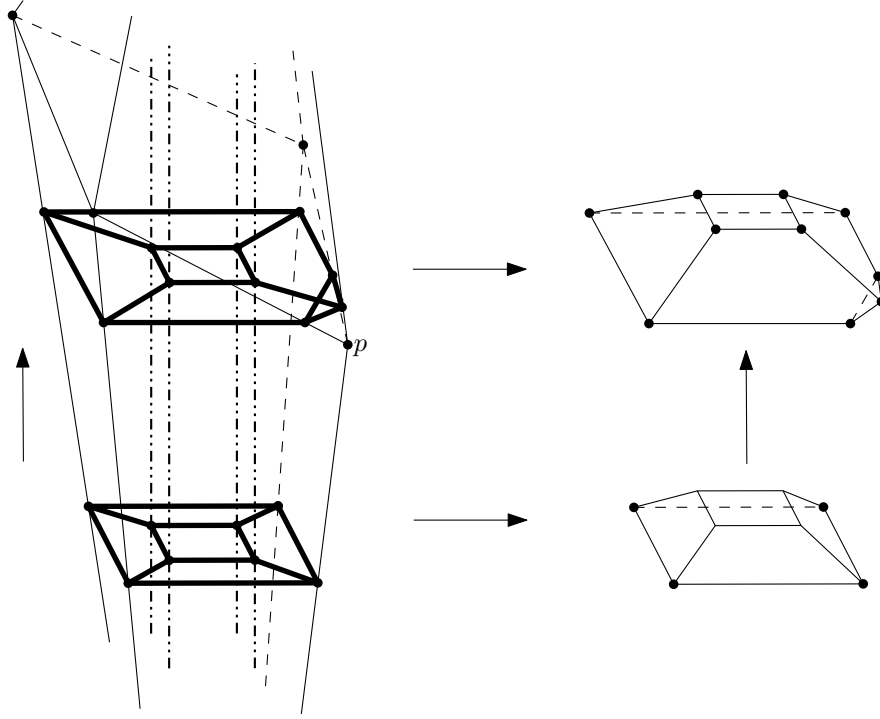
FIGURE 10. A commutative diagram

Indeed the LHS of Eq. (4.6) gives

$$\frac{\iota\left(\alpha(pq_i) \wedge \alpha(pq_k)\right)}{\langle \xi, \alpha(pq_i)\rangle} - \frac{\iota\left(\alpha(pq_j) \wedge \alpha(pq_k)\right)}{\langle \xi, \alpha(pq_j)\rangle}$$

$$= \frac{\langle \xi, \alpha(pq_k)\rangle}{\langle \xi, \alpha(pq_j)\rangle}\alpha(pq_j) - \frac{\langle \xi, \alpha(pq_k)\rangle}{\langle \xi, \alpha(pq_i)\rangle}\alpha(pq_i)$$

$$= m_{kj} \cdot \left(\alpha(pq_j) - \frac{\langle \xi, \alpha(pq_j)\rangle}{\langle \xi, \alpha(pq_i)\rangle}\alpha(pq_i)\right)$$

which yields the RHS. The verification of Eq. (4.5) is left to the reader.

Note that any function $A\colon E_{c_1}^{pq_1} \to \mathbb{R}^{d-1}$ that maps $E_{c_1}^{pq_1} = \{Q_{12}, Q_{13}, \ldots, Q_{1d}\}$ onto a basis extends to a generalized d-independent axial function $\hat{A}\colon E_{c_1} \to \mathbb{R}^{d-1}$ via the relations in Eqs. (4.5) and (4.6). Moreover $\alpha_{c_1}^d = p \circ \hat{A}$ where $p\colon \mathbb{R}^{d-1} \to \mathbb{R}^{n-1}$ is the surjective map mapping the basis vector $A(Q_{1i})$ to the vector $\alpha(Q_{1i})$.

$\square$

We are now in a position to explicitly write down a proof of Theorem 1.

Proof of Theorem 1. Assume $(\Gamma, A, \theta, \lambda)$ is a total lift of $(\Gamma, \alpha, \theta, \lambda)$, and let Q be a 2-face with normal edges $N_Q \subset E_\Gamma$. Fix a base point p_0 in Q and let $\gamma^Q \colon p_0 \to p_1 \cdots \to p_m \to p_0$ denote the loop in Q based at p_0. Note that the d-independence of A implies that Q is actually a 2-slice for some 2-dimensional subspace $H \subset \mathbb{R}^d$. Now for any edge $e \in N_Q^{p_0}$, the subspace spanned by $A(e)$ and H must necessarily contain the vector $A\left(K_{\gamma^Q}^\perp(e)\right)$. Hence by d-independence of A, we must have $K_{\gamma_Q}^\perp(e) = e$, hence Q has trivial normal holonomy. To compute the normal holonomy connection number for an oriented edge $e' \in N_Q$, choose and fix a covector $\eta \in \left(\mathbb{R}^d\right)^*$ vanishing on the subspace H, but not vanishing on $A(e')$. Then for $e \in Q$ at $i(e')$ we have

$$(4.7) \quad \alpha(e') - \frac{\eta(\alpha(e'))}{\eta(\alpha(\theta_e(e')))} \alpha(\theta_e(e')) \ \in H \cap \mathrm{span}\,\{A(e), A(e')\} = \mathrm{span}\,\{A(e)\}.$$

By d-independence of A, we deduce that $\lambda_e(e') = \dfrac{\eta(\alpha(e'))}{\eta(\alpha(\theta_e(e')))}$, and thus readily conclude that $\left|K_{\gamma^Q}^\perp\right| = 1$.

Conversely, let $(\Gamma, \alpha, \theta, \lambda)$ be reducible d-valent 1-skeleton in $\mathbb{R}^n$ whose 2-faces are all level with trivial normal holonomy. Then by Lemma 2, there is a $(d + 1)$-valent 1-skeleton $(\hat{\Gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})$ in $\mathbb{R}^n \times \mathbb{R}$ that is reducible via the covector $\hat{\xi} = (\xi, \mathbf{1}) \in (\mathbb{R}^n \times \mathbb{R})^*$ and that also satisfies (†). Moreover Lemma 2 tells us that $(\Gamma, \alpha, \theta, \lambda) \equiv (\hat{\Gamma}_c, \pi \circ \hat{\alpha}_c^d, \hat{\theta}_c^d, \hat{\lambda}_c^d)$, for some linear isomorphism $\pi \colon W_{\hat{\xi}} \to \mathbb{R}^n$. By Lemma 4 $(\hat{\Gamma}_c, \hat{\alpha}_c^d, \hat{\theta}_c^d, \hat{\lambda}_c^d)$ must have a total lift, and hence so must $(\hat{\Gamma}_c, \pi \circ \hat{\alpha}_c^d, \hat{\theta}_c^d, \hat{\lambda}_c^d)$. Thus $(\Gamma, \alpha, \theta, \lambda)$ also has a total lift.

This establishes Theorem 1.   □

We can now prove Corollary 1, which characterizes 1-skeleta of projected simple polytopes.

Corollary 1. *A d-valent 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ is the 1-skeleton of a projected simple polytope if and only if $(\Gamma, \alpha, \theta, \lambda)$ is reducible, all its 2-faces are level with trivial normal holonomy, and it admits an embedding $F \colon V_\Gamma \to \mathbb{R}^n$.*

Proof. Assume $(\Gamma, \alpha, \theta, \lambda)$ is the 1-skeleton of a projected simple d-polytope $S \subset \mathbb{R}^d$ via projection $p \colon \mathbb{R}^d \to \mathbb{R}^n$. Then the 1-skeleton of S, $(\Gamma, A, \theta, \lambda)$, is a total lift of $(\Gamma, \alpha, \theta, \lambda)$ with respect to $p \colon \mathbb{R}^d \to \mathbb{R}^n$. Thus by Theorem 1 all 2-faces of $(\Gamma, \alpha, \theta, \lambda)$ are level with trivial normal holonomy. Moreover the natural embedding

$$V_S \xrightarrow{\quad F \quad} \mathbb{R}^d$$

$$v \longmapsto \vec{v}$$

of $(\Gamma, A, \theta, \lambda)$ composes with the projection p to give an embedding $f :=$ $p \circ F \colon E_\Gamma \to \mathbb{R}^n$ of $(\Gamma, \alpha, \theta, \lambda)$.

Conversely assume that $(\Gamma, \alpha, \theta, \lambda)$ is a d-valent reducible 1-skeleton in $\mathbb{R}^n$ whose 2-faces are level with trivial normal holonomy, and that admits an embedding $f \colon V_\Gamma \to \mathbb{R}^n$. Then by Theorem 1, $(\Gamma, \alpha, \theta, \lambda)$ has a total lift $(\Gamma, A, \theta, \lambda) \subset \mathbb{R}^d$ with respect to some projection $p \colon \mathbb{R}^d \to \mathbb{R}^n$. Since $(\Gamma, A, \theta, \lambda)$ is a d-valent d-independent noncyclic 1-skeleton, a result of Guillemin and Zara [4, Theorem 2.4.4] implies that there is a map $F \colon V_\Gamma \to \mathbb{R}^d$ such that $f = F \circ p$. By linearity of p, F must be an embedding of $(\Gamma, A, \theta, \lambda)$. Hence $(\Gamma, A, \theta, \lambda)$, a d-valent d-independent noncyclic 1-skeleton with an embedding, is the 1-skeleton of a simple d-polytope.   □

5. Concluding Remarks

Besides Corollary 1, Theorem 1 has the following rather interesting consequence: Any d-valent 4-independent non-cyclic 1-skeleton is the projection of a toral 1-skeleton. In other words, the 2-faces of 4-independent reducible 1-skeleta are necessarily level with trivial normal holonomy. Indeed fix any 2-face Q of a 4-independent reducible 1-skeleton $(\Gamma, \alpha, \theta, \lambda)$ in $\mathbb{R}^n$. Fix any vertex $p \in Q$, let $e_1, e_2 \in E_Q^p$ denote the oriented edges at p spanning Q, and let γ_Q be a loop in Q based at p. Then for any $e' \in N_Q^p$, the vectors $\alpha(e_1)$, $\alpha(e_2)$, $\alpha(e')$ and $\alpha(K_{\gamma_Q}(e'))$ must span the same 3-dimensional subspace. By 4-independence, we conclude that $K_{\gamma_Q}(e')$ must equal e', hence Q has trivial normal holonomy. Fix a nonzero covector $\eta \in (\mathbb{R}^n)^*$ that pairs to zero with $\alpha(e_i)$ for $i = 1, 2$, but nonzero with $\alpha(e)$ for $e \in N_Q^p$. Then 4-independence is enough to guarantee that the compatibility system λ must be

$$\lambda_e(e') = \frac{\langle \eta, \alpha(e') \rangle}{\langle \eta, \alpha(\theta_e(e')) \rangle}.$$

From this it is straight forward to verify that $\left| K_{\gamma_Q}^\perp \right| = 1$, hence that Q is level.

In contrast, there are definitely 3-independent noncyclic 1-skeleta that are not projections of toral 1-skeleta. For example, the reader can readily verify the 3-independent 1-skeleton in Figure 11 does not have a lift, since the 2-face $\gamma \colon u \to v \to w \to u$ does not have trivial normal holonomy:

$$K_\gamma^\perp \colon ux \mapsto vx \mapsto wy \mapsto uy.$$

It would be interesting to find an example of a toral 1-skeleton that is not noncyclic, i.e. that does not admit a polarization. An example of a non-shellable complete simplicial fan would certainly do. Indeed the vertex ordering on a polarized toral 1-skeleton coming from a compatible Morse
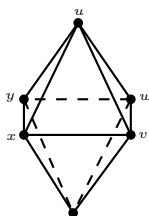
FIGURE 11. No Lift

function necessarily defines a shelling order for the corresponding facets of the fan.

The essence of Theorem 1 is that reducible 1-skeleta whose cross sections behave like simple polytopes are either simple polytopes or projections of simple polytopes. The dictionary between GKM manifolds and 1-skeleta seems to suggest that there may be an analogue of Theorem 1 in geometry. More precisely, is there an example of a symplectic $2d$-manifold with an effective Hamiltonian T^n action that is not a toric manifold itself, but whose reduced spaces (with respect to some fixed generic $S^1 \subset T^n$) are all toric manifolds?

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DEPT. OF MATH. AND COMP. SCI., ENDICOTT COLLEGE, BEVERLY, MA 01915
E-mail address: cmcdanie@endicott.edu